C3 Numerical Methods

1. June 2010 qu. 6
   (i) Show by calculation that the equation \( \tan^2 x - x - 2 = 0 \), where \( x \) is measured in radians, has a root between 1.0 and 1.1. \[3\]
   (ii) Use the iteration formula \( x_{n+1} = \tan^{-1} \left( \sqrt{2 + x_n} \right) \) with a suitable starting value to find this root correct to 5 decimal places. You should show the outcome of each step of the process. \[4\]
   (iii) Deduce a root of the equation \( \sec^2 2x - 2x - 3 = 0 \). \[3\]

2. Jan 2010 qu.3
   (i) Find, in simplified form, the exact value of \( \int_{0}^{20} \frac{60}{x} \, dx \). \[2\]
   (ii) Use Simpson’s rule with two strips to find an approximation to \( \int_{0}^{20} \frac{60}{x} \, dx \). \[3\]
   (iii) Use your answers to parts (i) and (ii) to show that \( \ln 2 \approx \frac{25}{36} \). \[2\]

3. Jan 2010 qu. 8
   (i) The curve \( y = \sqrt{x} \) can be transformed to the curve \( y = \sqrt{2x+3} \) by means of a stretch parallel to the \( y \)-axis followed by a translation. State the scale factor of the stretch and give details of the translation. \[3\]
   (ii) It is given that \( N \) is a positive integer. By sketching on a single diagram the graphs of \( y = \sqrt{2x+3} \) and \( y = \frac{N}{x^3} \), show that the equation \( \sqrt{2x+3} = \frac{N}{x^3} \) has exactly one real root. \[3\]
   (iii) A sequence \( x_1, x_2, x_3, \ldots \) has the property that \( x_{n+1} = N \left( \frac{1}{2} x_n + 3 \right)^{\frac{1}{6}} \). For certain values of \( x_1 \) and \( N \), it is given that the sequence converges to the root of the equation \( \sqrt{2x+3} = \frac{N}{x^3} \).
      (a) Find the value of the integer \( N \) for which the sequence converges to the value 1.9037 (correct to 4 decimal places). \[2\]
      (b) Find the value of the integer \( N \) for which, correct to 4 decimal places, \( x_3 = 2.6022 \) and \( x_4 = 2.6282 \). \[3\]

4. FP2 Jan 2010 qu 1 part i)
   It is given that \( f(x) = x^2 - \sin x \).
      (i) The iteration \( x_{n+1} = \sqrt{\sin x_n} \), with \( x_1 = 0.875 \), is to be used to find a real root, \( \alpha \), of the equation \( f(x) = 0 \). Find \( x_2, x_3 \) and \( x_4 \), giving the answers correct to 6 decimal places. \[2\]

5. June 2009 qu. 4
   It is given that \( \int_{3a}^{3a} (e^{3x} + e^x) \, dx = 100 \), where \( a \) is a positive constant.
      (i) Show that \( a = \frac{1}{9} \ln(300 + 3e^a - 2e^{3a}) \). \[5\]
      (ii) Use an iterative process, based on the equation in part (i), to find the value of \( a \) correct to 4 decimal places. Use a starting value of 0.6 and show the result of each step of the process.

6. June 2009 qu. 8
The diagram shows the curves \( y = \ln x \) and \( y = 2 \ln(x - 6) \). The curves meet at the point \( P \) which has \( x \)-coordinate \( a \). The shaded region is bounded by the curve \( y = 2 \ln(x - 6) \) and the lines \( x = a \) and \( y = 0 \).

(i) Give details of the pair of transformations which transforms the curve \( y = \ln x \) to the curve \( y = 2 \ln(x - 6) \). \( [3] \)

(ii) Solve an equation to find the value of \( a \). \( [4] \)

(iii) Use Simpson’s rule with two strips to find an approximation to the area of the shaded region. \( [3] \)

7. Jan 2009 qu. 2

(i) Use Simpson’s rule with four strips to find an approximation to \( \int_4^{12} \ln x \ dx \), giving your answer correct to 2 decimal places. \( [4] \)

(ii) Deduce an approximation to \( \int_4^{12} \ln(x^{10}) \ dx \). \( [1] \)

8. Jan 2009 qu. 6

The function \( f \) is defined for all real values of \( x \) by

\[
f(x) = \sqrt[3]{\frac{1}{2} x + 2}.
\]

The graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) meet at the point \( P \), and the graph of \( y = f^{-1}(x) \) meets the \( x \)-axis at \( Q \) (see diagram).

(i) Find an expression for \( f^{-1}(x) \) and determine the \( x \)-coordinate of the point \( Q \). \( [3] \)
(ii) State how the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) are related geometrically, and hence show that the \( x \)-coordinate of the point \( P \) is the root of the equation \( x = \frac{1}{2} \sqrt{1 - x + 2} \). [2]

(iii) Use an iterative process, based on the equation \( x = \frac{1}{2} \sqrt{1 - x + 2} \), to find the \( x \)-coordinate of \( P \), giving your answer correct to 2 decimal places. [4]

9. **FP2 Jan 2009 qu. 2 part i)**

It is given that \( \alpha \) is the only real root of the equation \( x^5 + 2x - 28 = 0 \) and that \( 1.8 < \alpha < 2 \).

(i) The iteration \( x_{n+1} = \frac{1}{3} \sqrt{28 - 2x_n} \), with \( x_1 = 1.9 \), is to be used to find \( \alpha \). Find the values of \( x_2 \), \( x_3 \) and \( x_4 \), giving the answers correct to 7 decimal places. [3]

10. **June 2008 qu. 4**

The gradient of the curve \( y = (2x^2 + 9)^{\frac{5}{2}} \) at the point \( P \) is 100.

(i) Show that the \( x \)-coordinate of \( P \) satisfies the equation \( x = \frac{3}{5} \left( 2x^2 + 9 \right)^{\frac{2}{3}} \). [3]

(ii) Show by calculation that the \( x \)-coordinate of \( P \) lies between 0.3 and 0.4. [3]

(iii) Use an iterative formula, based on the equation in part (i), to find the \( x \)-coordinate of \( P \) correct to 4 decimal places. You should show the result of each iteration. [3]

11. **Jan 2008 qu. 2**

The sequence defined by \( x_1 = 3 \), \( x_{n+1} = \frac{1}{3} \sqrt{31 - \frac{2}{5} x_n} \) converges to the number \( \alpha \).

(i) Find the value of \( \alpha \) correct to 3 decimal places, showing the result of each iteration. [3]

(ii) Find an equation of the form \( ax^3 + bx + c = 0 \), where \( a \), \( b \) and \( c \) are integers, which has \( \alpha \) as a root. [3]

12. **June 2007 qu. 6**

(i) Given that \( \int_0^a (6e^{2x} + x)dx = 42 \), show that \( a = \frac{1}{2} \ln(15 - \frac{1}{6} a^2) \). [5]

(ii) Use an iterative formula, based on the equation in part (i), to find the value of \( a \) correct to 3 decimal places. Use a starting value of 1 and show the result of each iteration. [4]

13. **Jan 2007 qu. 3**

(a) It is given that \( a \) and \( b \) are positive constants. By sketching graphs of \( y = x^5 \) and \( y = a - bx \) on the same diagram, show that the equation \( x^5 + bx - a = 0 \) has exactly one real root. [3]

(b) Use the iterative formula \( x_{n+1} = \frac{1}{2} \sqrt{53 - 2x_n} \), with a suitable starting value, to find the real root of the equation \( x^5 + 2x - 53 = 0 \). Show the result of each iteration, and give the root correct to 3 decimal places. [4]

14. **Jan 2007 qu. 8**
The diagram shows the curve with equation \( y = x^8 e^{-x^2} \). The curve has maximum points at \( P \) and \( Q \). The shaded region \( A \) is bounded by the curve, the line \( y = 0 \) and the line through \( Q \) parallel to the \( y \)-axis. The shaded region \( B \) is bounded by the curve and the line \( PQ \).

(i) Show by differentiation that the \( x \)-coordinate of \( Q \) is 2. 

(ii) Use Simpson’s rule with 4 strips to find an approximation to the area of region \( A \). Give your answer correct to 3 decimal places.

(iii) Deduce an approximation to the area of region \( B \).

15. June 2006 qu. 3

The equation \( 2x^3 + 4x - 35 = 0 \) has one real root.

(i) Show by calculation that this real root lies between 2 and 3.

(ii) Use the iterative formula 

\[ x_{n+1} = \sqrt[3]{17.5 - 2x_n} \]

with a suitable starting value, to find the real root of the equation \( 2x^3 + 4x - 35 = 0 \) correct to 2 decimal places. You should show the result of each iteration.

16. Jan 2006 qu. 7

The diagram shows the curve with equation \( y = \cos^{-1}x \).

(i) Sketch the curve with equation \( y = 3 \cos^{-1}(x-1) \), showing the coordinates of the points where the curve meets the axes.

(ii) By drawing an appropriate straight line on your sketch in part (i), show that the equation \( 3 \cos^{-1}(x-1) = x \) has exactly one root.

(iii) Show by calculation that the root of the equation \( 3 \cos^{-1}(x-1) = x \) lies between 1.8 and 1.9.

(iv) The sequence defined by \( x_1 = 2, \ x_{n+1} = 1 + \cos \left( \frac{1}{3} x_n \right) \)

converges to a number \( \alpha \). Find the value of \( \alpha \) correct to 2 decimal places and explain why \( \alpha \) is the root of the equation \( 3 \cos^{-1}(x-1) = x \).
17. **Jan 2006 qu. 8**

The diagram shows part of the curve \( y = \ln(5 - x^2) \) which meets the \( x \)-axis at the point \( P \) with coordinates \((2, 0)\). The tangent to the curve at \( P \) meets the \( y \)-axis at the point \( Q \). The region \( A \) is bounded by the curve and the lines \( x = 0 \) and \( y = 0 \). The region \( B \) is bounded by the curve and the lines \( PQ \) and \( x = 0 \).

(i) Find the equation of the tangent to the curve at \( P \). \([5]\)

(ii) Use Simpson’s Rule with four strips to find an approximation to the area of the region \( A \), giving your answer correct to 3 significant figures. \([4]\)

(iii) Deduce an approximation to the area of the region \( B \). \([2]\)

18. **June 2005 qu. 8**

The diagram shows part of each of the curves \( y = e^{\frac{1}{3} x} \) and \( y = \frac{3}{3} \sqrt{3x + 8} \). The curves meet, as shown in the diagram, at the point \( P \). The region \( R \), shaded in the diagram, is bounded by the two curves and by the \( y \)-axis.

(i) Show by calculation that the \( x \)-coordinate of \( P \) lies between 5.2 and 5.3. \([3]\)

(ii) Show that the \( x \)-coordinate of \( P \) satisfies the equation \( x = \frac{5}{3} \ln(3x + 8) \). \([2]\)

(iii) Use an iterative formula, based on the equation in part (ii), to find the \( x \)-coordinate of \( P \) correct to 2 decimal places. \([3]\)

(iv) Use integration, and your answer to part (iii), to find an approximate value of the area of the region \( R \). \([5]\)

19. **June 2005 qu. 4**

(a) The diagram shows the curve \( y = \frac{2}{\sqrt{x}} \).

The region \( R \), shaded in the diagram, is bounded by the curve and by the lines \( x = 1 \), \( x = 5 \) and \( y = 0 \). The region \( R \) is rotated completely about the \( x \)-axis.

Find the exact volume of the solid formed. \([4]\)

(b) Use Simpson’s rule, with 4 strips, to find an approximate value for

\[
\int_1^5 \sqrt{(x^2 + 1)} \, dx,
\]

giving your answer correct to 3 decimal places. \([4]\)