C3 Functions

1. **June 2010 qu.9**

   The functions \( f \) and \( g \) are defined for all real values of \( x \) by
   \[
   f(x) = 4x^2 - 12x \quad \text{and} \quad g(x) = ax + b,
   \]
   where \( a \) and \( b \) are non-zero constants.
   (i) Find the range of \( f \). \[3\]
   (ii) Explain why the function \( f \) has no inverse. \[2\]
   (iii) Given that \( g^{-1}(x) = g(x) \) for all values of \( x \), show that \( a = -1 \). \[4\]
   (iv) Given further that \( gf(x) < 5 \) for all values of \( x \), find the set of possible values of \( b \). \[4\]

2. **Jan 2010 qu.4**

   The function \( f \) is defined for all real values of \( x \) by
   \[
   f(x) = 2 - \sqrt{x+1}.
   \]
   The diagram shows the graph of \( y = f(x) \).
   (i) Evaluate \( f(f(-126)) \). \[2\]
   (ii) Find the set of values of \( x \) for which \( f(x) = |f(x)| \). \[2\]
   (iii) Find an expression for \( f^{-1}(x) \). \[3\]
   (iv) State how the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) are related geometrically. \[1\]

3. **June 2009 qu.5**

   The functions \( f \) and \( g \) are defined for all real values of \( x \) by
   \[
   f(x) = 3x - 2 \quad \text{and} \quad g(x) = 3x + 7.
   \]
   Find the exact coordinates of the point at which
   (i) the graph of \( y = fg(x) \) meets the \( x \)-axis, \[3\]
   (ii) the graph of \( y = g(x) \) meets the graph of \( y = g^{-1}(x) \), \[3\]
   (iii) the graph of \( y = |f(x)| \) meets the graph of \( y = |g(x)| \). \[4\]

4. **June 2009 qu.8**

   The diagram shows the curves \( y = \ln x \) and \( y = 2 \ln(x - 6) \). The curves meet at the point \( P \) which has \( x \)-coordinate \( a \). The shaded region is bounded by the curve \( y = 2 \ln(x - 6) \) and the lines \( x = a \) and \( y = 0 \).
   (i) Give details of the pair of transformations which transforms the curve \( y = \ln x \) to the curve \( y = 2 \ln(x - 6) \). \[3\]
   (ii) Solve an equation to find the value of \( a \). \[4\]
5. Jan 2009 qu.6

The function \( f \) is defined for all real values of \( x \) by

\[
f(x) = \sqrt{\frac{1}{2}x + 2}.
\]

The graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) meet at the point \( P \), and the graph of \( y = f^{-1}(x) \) meets the \( x \)-axis at \( Q \) (see diagram).

(i) Find an expression for \( f^{-1}(x) \) and determine the \( x \)-coordinate of the point \( Q \). [3]

(ii) State how the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) are related geometrically, and hence show that the \( x \)-coordinate of the point \( P \) is the root of the equation

\[
x = \sqrt{\frac{1}{2}x + 2}.
\]

6. Jan 2009 qu.7

The diagram shows the curve \( y = e^{kx} - a \), where \( k \) and \( a \) are constants.

(i) Give details of the pair of transformations which transforms the curve \( y = e^x \) to the curve \( y = e^{kx} - a \). [3]

(ii) Sketch the curve \( y = \left| e^{kx} - a \right| \). [2]

(iii) Given that the curve \( y = \left| e^{kx} - a \right| \) passes through the points \((0, 13)\) and \((\ln 3, 13)\), find the values of \( k \) and \( a \). [4]

7. June 2008 qu.1

Find the exact solutions of the equation

\[
\left| 4x - 5 \right| = \left| 3x - 5 \right|.
\]
8. June 2008 qu.2

The diagram shows the graph of \( y = f(x) \). It is given that \( f(-3) = 0 \) and \( f(0) = 2 \). Sketch, on separate diagrams, the following graphs, indicating in each case the coordinates of the points where the graph crosses the axes:

(i) \( y = f^{-1}(x) \), \[2\]

(ii) \( y = -2f(x) \). \[3\]

9. June 2008 qu.7

It is claimed that the number of plants of a certain species in a particular locality is doubling every 9 years. The number of plants now is 42. The number of plants is treated as a continuous variable and is denoted by \( N \). The number of years from now is denoted by \( t \).

(i) Two equivalent expressions giving \( N \) in terms of \( t \) are

\[ N = A \times 2^{kt} \quad \text{and} \quad N = Ae^{mt}. \]

Determine the value of each of the constants \( A \), \( k \) and \( m \). \[4\]

(ii) Find the value of \( t \) for which \( N = 100 \), giving your answer correct to 3 significant figures. \[2\]

(iii) Find the rate at which the number of plants will be increasing at a time 35 years from now. \[3\]

10. Jan 2008 qu.1

Functions \( f \) and \( g \) are defined for all real values of \( x \) by \( f(x) = x^3 + 4 \) and \( g(x) = 2x - 5 \).

Evaluate

(i) \( fg(1) \), \[2\]

(ii) \( f^{-1}(12) \). \[3\]

11. Jan 2008 qu.6

The diagram shows the graph of \( y = -\sin^{-1}(x - 1) \).

(i) Give details of the pair of geometrical transformations which transforms the graph of \( y = -\sin^{-1}(x - 1) \) to the graph of \( y = \sin^{-1}x \). \[3\]

(ii) Sketch the graph of \( y = \left| -\sin^{-1}(x - 1) \right| \). \[2\]

(iii) Find the exact solutions of the equation \( \left| -\sin^{-1}(x - 1) \right| = \frac{1}{3}\pi \). \[3\]
12. **June 2007 qu.2**  
Solve the inequality $|4x - 3| < |2x + 1|$.  

13. **June 2007 qu.3**  
The function $f$ is defined for all non-negative values of $x$ by $f(x) = 3 + \sqrt{x}$.  
(i) Evaluate $f(169)$.  
(ii) Find an expression for $f^{-1}(x)$ in terms of $x$.  
(iii) On a single diagram sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, indicating how the two graphs are related.  

14. **June 2007 qu.5**  
A substance is decaying in such a way that its mass, $m$ kg, at a time $t$ years from now is given by the formula $m = 240e^{-0.04t}$.  
(i) Find the time taken for the substance to halve its mass.  
(ii) Find the value of $t$ for which the mass is decreasing at a rate of 2.1 kg per year.  

15. **Jan 2007 qu.9**  
Functions $f$ and $g$ are defined by $f(x) = 2 \sin x$ for $\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$,  
$g(x) = 4 - 2x^2$ for $x \leq \;$.  
(i) State the range of $f$ and the range of $g$.  
(ii) Show that $gf(0.5) = 2.16$, correct to 3 significant figures, and explain why $fg(0.5)$ is not defined.  
(iii) Find the set of values of $x$ for which $f^{-1}g(x)$ is not defined.  

16. **June 2006 qu.2**  
Solve the inequality $|2x - 3| < |x + 1|$.  

17. **June 2006 qu.6**  

The diagram shows the graph of $y = f(x)$, where $f(x) = 2 - x^2$, $x \leq 0$.  
(i) Evaluate $ff(-3)$.  
(ii) Find an expression for $f^{-1}(x)$.  
(iii) Sketch the graph of $y = f^{-1}(x)$. Indicate the coordinates of the points where the graph meets the axes.
18. **Jan 2006 qu.4**

The function \( f \) is defined by \( f(x) = 2 - \sqrt{x} \) for \( x \geq 0 \). The graph of \( y = f(x) \) is shown above.

(i) State the range of \( f \). [1]

(ii) Find the value of \( f(f(4)) \). [2]

(iii) Given that the equation \(|f(x)| = k\) has two distinct roots, determine the possible values of the constant \( k \). [2]

19. **June 2005 qu.1**

The function \( f \) is defined for all real values of \( x \) by \( f(x) = 10 - (x + 3)^2 \).

(i) State the range of \( f \). [1]

(ii) Find the value of \( f(f(-1)) \). [3]

20. **June 2005 qu.2**

Find the exact solutions of the equation \(|6x - 1| = |x - 1|\). [4]

21. **June 2005 qu.9**

The function \( f \) is defined by \( f(x) = \sqrt{mx + 7} - 4 \), where \( x \geq -\frac{7}{m} \) and \( m \) is a positive constant.

The diagram shows the curve \( y = f(x) \).

(i) A sequence of transformations maps the curve \( y = \sqrt{x} \) to the curve \( y = f(x) \). Give details of these transformations. [4]

(ii) Explain how you can tell that \( f \) is a one–one function and find an expression for \( f^{-1}(x) \). [4]

(iii) It is given that the curves \( y = f(x) \) and \( y = f^{-1}(x) \) do not meet. Explain how it can be deduced that neither curve meets the line \( y = x \), and hence determine the set of possible values of \( m \). [5]