1 A cup of water is cooling. Its initial temperature is 100°C. After 3 minutes, its temperature is 80°C.

(i) Given that \( T = 25 + ae^{-kt} \), where \( T \) is the temperature in °C, \( t \) is the time in minutes and \( a \) and \( k \) are constants, find the values of \( a \) and \( k \). [5]

(ii) What is the temperature of the water

(A) after 5 minutes,

(B) in the long term? [3]

2 A population is \( P \) million at time \( t \) years. \( P \) is modelled by the equation

\[ P = 5 + ae^{-bt}, \]

where \( a \) and \( b \) are constants.

The population is initially 8 million, and declines to 6 million after 1 year.

(i) Use this information to calculate the values of \( a \) and \( b \), giving \( b \) correct to 3 significant figures. [5]

(ii) What is the long-term population predicted by the model? [1]

3 (i) Express \( 2\ln x + \ln 3 \) as a single logarithm. [2]

(ii) Hence, given that \( x \) satisfies the equation

\[ 2\ln x + \ln 3 = \ln (5x + 2), \]

show that \( x \) is a root of the quadratic equation \( 3x^2 - 5x - 2 = 0 \). [2]

(iii) Solve this quadratic equation, explaining why only one root is a valid solution of

\[ 2\ln x + \ln 3 = \ln (5x + 2). \] [3]
4 The mass $M$ kg of a radioactive material is modelled by the equation

$$M = M_0 e^{-kt},$$

where $M_0$ is the initial mass, $t$ is the time in years, and $k$ is a constant which measures the rate of radioactive decay.

(i) Sketch the graph of $M$ against $t$. [2]

(ii) For Carbon 14, $k = 0.000121$. Verify that after 5730 years the mass $M$ has reduced to approximately half the initial mass. [2]

The half-life of a radioactive material is the time taken for its mass to reduce to exactly half the initial mass.

(iii) Show that, in general, the half-life $T$ is given by $T = \frac{\ln 2}{k}$. [3]

(iv) Hence find the half-life of Plutonium 239, given that for this material $k = 2.88 \times 10^{-5}$. [1]

5 The temperature $T$ °C of a liquid at time $t$ minutes is given by the equation

$$T = 30 + 20e^{-0.05t}, \quad \text{for } t \geq 0.$$

Write down the initial temperature of the liquid, and find the initial rate of change of temperature.

Find the time at which the temperature is 40 °C. [6]
Fig. 8 shows a sketch of part of the curve $y = x \sin 2x$, where $x$ is in radians.

The curve crosses the $x$-axis at the point $P$. The tangent to the curve at $P$ crosses the $y$-axis at $Q$.

(i) Find $\frac{dy}{dx}$. Hence show that the $x$-coordinates of the turning points of the curve satisfy the equation $\tan 2x + 2x = 0$.

(ii) Find, in terms of $\pi$, the $x$-coordinate of the point $P$.

Show that the tangent $PQ$ has equation $2\pi x + 2y = \pi^2$.

Find the exact coordinates of $Q$.

(iii) Show that the exact value of the area shaded in Fig. 8 is $\frac{1}{3} \pi(\pi^2 - 2)$.