## Question

Given the function \( f(x) = \frac{x(2x-2) - (x-2)^2}{x^2} \), find the derivative \( f'(x) \) and determine the critical points by setting \( f'(x) = 0 \). Use the second derivative test to classify each critical point as a maximum, minimum, or neither.

### Solution

1. **Find the derivative**

   \[
   f'(x) = \frac{d}{dx} \left( \frac{x(2x-2) - (x-2)^2}{x^2} \right)
   \]

2. **Simplify**

   \[
   f'(x) = \frac{x(2x-2) - (x-2)^2}{x^2} = \frac{2x^2 - 4x - x^2 + 4x - 4}{x^2} = \frac{x^2 - 4}{x^2} = 1 - 4/x^2
   \]

3. **OR**

   \[
   f(x) = \frac{(x^2 - 4x + 4)}{x} = x - 4 + 4/x
   \]

4. **Differentiate**

   \[
   f'(x) = 1 - 4/x^2
   \]

5. **Simplify correctly**

   \[
   f''(x) = \frac{8}{x^3}
   \]

6. **Critical Points**

   \[
   f'(x) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2
   \]

   So, \( Q \) is \((-2, -8)\) and \( (2, 0) \) is another point.

7. **Second Derivative Test**

   \[
   f''(-2) = \frac{8}{(-2)^3} = -1 < 0 \Rightarrow \text{maximum at } (-2, -8)
   \]

   \[
   f''(2) = \frac{8}{2^3} = 1 > 0 \Rightarrow \text{minimum at } (2, 0)
   \]

### Notes

- Use the quotient (or product) rule for differentiation.
- Condone sign errors only if the final working is correct.
- Correct the exponent, and condone missing brackets.
- Simplify the derivative correctly.
- Ensure that the second derivative test is used correctly.
- State the nature of the stationary points (maximum, minimum) correctly.

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**NB AG**

- Any error in the working should be corrected.
- Any correct use of brackets is highlighted.
- The working must be neat and clear.

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<th>Question</th>
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<tr>
<td>(ii)</td>
<td>( f(1) = (-1)^{\frac{3}{2}}/1 = 1 ) ( f(4) = (2)^{\frac{3}{2}}/4 = 1 )</td>
<td>B1</td>
<td>verifying ( f(1) = 1 ) and ( f(4) = 1 ) or ( (x-2)^2 = x \Rightarrow x^2 - 5x + 4 = 0 ) ( (x-1)(x-4) = 0 ), ( x = 1, 4 )</td>
</tr>
<tr>
<td>[ \int_1^4 (x-2)^2 , dx = \int_1^4 (x-4 + 4/x) , dx ]</td>
<td>M1</td>
<td>expanding bracket and dividing each term by ( x ) ( \text{3 terms: } x - 4/x \text{ is M0} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = \left[ x^2/2 - 4x + 4 \ln x \right]_1^4 )</td>
<td>A1</td>
<td>( x^2/2 - 4x + 4 \ln x )</td>
</tr>
<tr>
<td></td>
<td>( = (8 - 16 + 4\ln 4) - (\frac{1}{2} - 4 + 4\ln 1) )</td>
<td>A1</td>
<td>( x^2/2 - 4x + 4 \ln x )</td>
</tr>
<tr>
<td></td>
<td>( = 4\ln 4 - 4\frac{1}{2} )</td>
<td>A1 cao</td>
<td>o.e. but must combine numerical terms and evaluate ( \ln 1 ) – mark final ans</td>
</tr>
<tr>
<td></td>
<td>Area enclosed = rectangle – curve ( = 3 \times 1 - (4\ln 4 - 4\frac{1}{2}) = 7\frac{1}{2} - 4\ln 4 )</td>
<td>M1</td>
<td>soi</td>
</tr>
<tr>
<td></td>
<td>[ \text{or} ]</td>
<td>A1 cao</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Area = ( \int_1^4 [1 - \left( \frac{x-2}{x} \right)^2] , dx )</td>
<td>M1</td>
<td>no need to have limits</td>
</tr>
<tr>
<td></td>
<td>( = \int_1^4 (5-x - 4/x) , dx )</td>
<td>M1</td>
<td>expanding bracket and dividing each term by ( x ) ( \text{must be 3 terms in } (x-2)^2 ) expansion</td>
</tr>
<tr>
<td></td>
<td>( = \left[ 5x - x^2/2 - 4 \ln x \right]_1^4 )</td>
<td>A1</td>
<td>( 5x - x^2/2 - 4 \ln x )</td>
</tr>
<tr>
<td></td>
<td>( = 20 - 8 - 4 \ln 4 - (5 - \frac{1}{2} - 4\ln 1) )</td>
<td>A1</td>
<td>o.e. but must combine numerical terms and evaluate ( \ln 1 ) – mark final ans</td>
</tr>
<tr>
<td></td>
<td>( = 7\frac{1}{2} - 4\ln 4 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(iii)</td>
<td>[ g(x) = f(x+1) - 1 ] ( = \frac{(x+1-2)^3}{x+1} - 1 )</td>
<td>M1</td>
<td>soi [may not be stated]</td>
</tr>
<tr>
<td></td>
<td>( = \frac{x^3 - 2x + 1 - x - 1}{x+1} = \frac{x^2 - 3x}{x+1} )</td>
<td>A1</td>
<td>correctly simplified – not from wrong working</td>
</tr>
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</table>
| | | A1 | |}

\( \text{NB AG} \)

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<td>1 (iv)</td>
<td>Area is the same as that found in part (ii)</td>
<td>M1</td>
<td>award M1 for ± ans to 8(ii) (unless zero)</td>
</tr>
<tr>
<td></td>
<td>[4\ln 4 - 7\frac{1}{2}]</td>
<td>A1cao</td>
<td>need not justify the change of sign</td>
</tr>
</tbody>
</table>

| 2 (i)    | \(xe^{-2x} = mx\)  
\(\Rightarrow e^{-2x} = m\)  
\(\Rightarrow -2x = \ln m\)  
\(\Rightarrow x = -\frac{1}{2} \ln m \quad (*)\)  
\(|\text{or}\)  
If \(x = -\frac{1}{2} \ln m, y = -\frac{1}{2} \ln m \times e^{\ln m}\)  
\(= -\frac{1}{2} \ln m \times m\)  
so P lies on \(y = mx\) | M1 | may be implied from 2\(^{nd}\) line |
|          | \(\text{dividing by } x, \text{ or subtracting } \ln x\) | M1 | o.e. e.g. \([\ln x] - 2x = \ln m + [\ln x]\) |
|          | \(\text{or factorising: } x(e^{-2x} - m) = 0\) | NB AG | or factorising: \(x(e^{-2x} - m) = 0\) |
|          | \(|\text{or}\)  
If \(x = -\frac{1}{2} \ln m\), \(y = -\frac{1}{2} \ln m \times e^{\ln m}\)  
\(= -\frac{1}{2} \ln m \times m\)  
so P lies on \(y = mx\) | A1 | substituting correctly |
|          | \(= -\frac{1}{2} \ln m \times m\) | A1 | |
|          | | A1 | |
|          | | [3] | |
| 2 (ii)   | let \(u = x\), \(u' = 1\), \(v = e^{-2x}\), \(v' = -2e^{-2x}\)  
d\(y/dx = e^{-2x} - 2xe^{-2x}\)  
\(= e^{-2\left(-\frac{1}{2}\ln m\right)} - 2\left(-\frac{1}{2}\ln m\right)e^{-2\left(-\frac{1}{2}\ln m\right)}\)  
\(= e^{\ln m} + e^{\ln m} \ln m \quad [= m + m \ln m]\) | M1* | product rule consistent with their derivs |
<p>|          | | A1 | o.e. correct expression |
|          | | M1dep | subst (x = -\frac{1}{2} \ln m) into their deriv dep |
|          | | M1* | |
|          | | A1cao | condone (e^{\ln m}) not simplified |
|          | | [4] | but not (-2\left(-\frac{1}{2} \ln m\right)), but mark final ans |</p>
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| 2 (iii)  | $m + m \ln m = -m$
$\Rightarrow \quad \ln m = -2$
$\Rightarrow \quad m = e^{-2}$
**or**
$y + \frac{1}{2} m \ln m = m(1 + \ln m)(x + \frac{1}{2} \ln m) = -1$,
$\Rightarrow 1 + \ln m = -1, \ln m = -2, m = e^{-2}$
At P, $x = 1$
$\Rightarrow \quad y = e^{-2}$
| M1 | their gradient from (ii) = $-m$ | A1 **NB AG**
| | for fully correct methods finding $x$-intercept of equation of tangent and equating to $-\ln m$ | B2
| | isw approximations | B1
| | not $e^{-2} \times 1$ | |
| (iv) Area under curve = $\int_{0}^{1} xe^{-2x} dx$
$u = x, \quad u' = 1, \quad v' = e^{-2x}, \quad v = -\frac{1}{2} e^{-2x}$
$= \left[-\frac{1}{2} xe^{-2x}\right]_{0}^{1} + \int_{0}^{1} \frac{1}{2} e^{-2x} dx$
$= \left[-\frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x}\right]_{0}^{1}$
$= (-\frac{1}{2} e^{-2} - \frac{1}{4} e^{-2}) - (0 - \frac{1}{4} e^{0})$
$\left[= \frac{1}{4} - \frac{1}{4} e^{-2}\right]$ 
| M1 | parts, condone $v = k e^{-2x}$, provided it is used consistently in their parts formula | A1
| | ignore limits until 3rd A1 | A1
| | fit their $v$ | A1
| | $-\frac{1}{2} xe^{-2x} - \frac{1}{4} e^{-2x}$ o.e. | A1
| | correct expression | A1
| | need not be simplified | A1
| Area of triangle = $\frac{1}{2}$ base $\times$ height
$= \frac{1}{2} \times 1 \times e^{-2}$
So area enclosed = $\frac{1}{4} - 5e^{-2}/4$
| M1 | fit their $1, e^{-2}$ or $[e^{-2}x^2/2]$ | A1
| | o.e. using isosceles triangle | A1cao
| | M1 may be implied from 0.067... | A1
| | isw | A7
### 3(i)

\[
\int_0^1 \frac{x^3}{1+x} \, dx \quad \text{let } u = 1 + x, \ du = dx
\]

when \( x = 0, u = 1, \) when \( x = 1, u = 2 \)

\[
= \int_1^2 \frac{(u-1)^3}{u} \, du
\]

\[
= \int_1^2 (u^3 - 3u^2 + 3u - 1) \, du
\]

\[
= \left[ \frac{u^4}{4} - \frac{u^3}{2} + \frac{u^2}{2} - u \right]_1^2
\]

\[
= \frac{1}{4} - \frac{3}{2} + 1 - 2 - \left( \frac{1}{4} - \frac{1}{2} + 1 - 1 \right)
\]

\[
= \frac{2}{3} - \frac{5}{18}
\]

**seen anywhere, e.g. in new limits**

**B1**
- \( a = 1, b = 2 \)
- \( (u-1)^3/u \)

**B1**
- expanding (correctly)

**A1 dep**
- \( \text{dep } du = \text{dx (o.e.) } AG \)

**B1**
- \( \left[ \frac{1}{3} \frac{3}{2} u^2 - 3u - \ln u \right] \)

**M1**
- substituting correct limits dep integrated

**A1 cao**
- must be exact – must be 5/6

**must have evaluated \( \ln 1 = 0 \)**

### 3(ii)

\[
y = x^2 \ln(1 + x)
\]

\[
\Rightarrow \quad \frac{dy}{dx} = 2x \cdot \frac{1}{1+x} + 2x \cdot \ln(1+x)
\]

\[
= \frac{x^2}{1+x} + 2x \ln(1+x)
\]

When \( x = 0, \frac{dy}{dx} = 0 + 0 \cdot \ln 1 = 0 \)

(⇒ Origin is a stationary point)

**M1**
- Product rule

**B1**
- \( \frac{d}{dx} \ln(1 + x) = \frac{1}{1 + x} \)

**A1**
- \( \text{cao (oe) mark final ans} \)

**or \( \frac{d}{dx} (\ln u) = \frac{1}{u} \text{ where } u = 1 + x \)**

**M1**
- substituting \( x = 0 \) into correct deriv

**A1 cao**
- **www**

**when \( x = 0, \frac{dy}{dx} = 0 \) with no evidence of substituting M1A0 but condone missing bracket in \( \ln(1+x) \)**

### 3(iii)

\[
A = \int_0^1 x^2 \ln(1 + x) \, dx
\]

let \( u = \ln(1 + x), \ dv/dx = x^2 \)

\[
\frac{du}{dx} = \frac{1}{1+x}, \quad v = \frac{x^3}{3}
\]

\[
\Rightarrow \quad A = \left[ \frac{x^3}{3} \ln(1 + x) \right]_0^1 - \int_0^1 \frac{x^3}{3+1} \, dx
\]

\[
= \frac{1}{3} \ln 2 - \left( \frac{5}{18} - \frac{1}{3} \ln 2 \right)
\]

\[
= \frac{1}{3} \ln 2 - \frac{5}{18} + \frac{1}{3} \ln 2
\]

\[
= \frac{2}{3} \ln 2 - \frac{5}{18}
\]

**B1**
- Correct integral and limits

**M1**
- parts correct

**B1**
- \( \frac{1}{3} \ln 2 - ... \)

**B1 ft**
- \( ... - 1/3 \) (result from part (i))

**A1**
- \( \text{cao} \)

**condone no dx, limits (and integral) can be implied by subsequent work**

**u, }du/dx, \ dv/dx and v all correct (oe)**

**condone missing brackets**

**condone missing bracket, can re-work from scratch**

**oe e.g.** \( \frac{12 \ln 2 - 5}{18} - \frac{1}{3} \ln 4 - \frac{5}{18} \) etc \ but must have evaluated \( \ln 1 = 0 \)

**Must combine the two \( \ln \) terms**