Edexcel M3 June 2016 Model Answers

1. (a)

$$
\begin{aligned}
v &= \frac{12}{x+3} \\
\frac{dv}{dx} &= -\frac{12}{(x+3)^2} \\
a &= v\frac{dv}{dx} \\
&= -\frac{144}{(x+3)^3} \\
F &= ma \\
&= 0.5\left(-\frac{144}{(x+3)^3}\right) \\
&= -\frac{72}{(x+3)^3} \\
F(x=3) &= -\frac{72}{(3+3)^3} \\
&= -\frac{1}{3} \\
\therefore |F| &= \tfrac{1}{3}\,\text{N}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{12}{x+3} \\
\int (x+3)\,dx &= 12\int dt \\
\tfrac{1}{2}x^2 + 3x &= 12t + C \\
\tfrac{1}{2}(4)^2 + 3(4) &= 12(2) + C \\
\therefore C &= -4 \\
\therefore \tfrac{1}{2}x^2 + 3x &= 12t - 4 \\
\tfrac{1}{2}(10)^2 + 3(10) &= 12t - 4 \\
12t &= 84 \\
\therefore t &= 7\,\text{s}
\end{aligned}
$$

2.

$$
\begin{aligned}
\overline{x} &= \frac{\int_a^b xy\,dx}{\int_a^b y\,dx} \\
&= \frac{\int_0^9 x(6-\frac{2}{3}x)\,dx}{\int_0^9 (6-\frac{2}{3}x)\,dx} \\
&= \frac{\int_0^9 (6x-\frac{2}{3}x^2)\,dx}{\int_0^9 (6-\frac{2}{3}x)\,dx} \\
&= \frac{\left(3x^2-\frac{2}{9}x^3\right)\Big|_0^9}{\left(6x-\frac{1}{3}x^2\right)\Big|_0^9} \\
&= \frac{3(9)^2-2(9)^2}{6(9)-\frac{1}{3}(9)} \\
&= 3\,\text{cm}
\end{aligned}
$$

3. $k = \frac{\lambda}{\ell} = \frac{14.7}{1.5} = 9.8\,\text{N m}^{-1}$

(a)

$$
\begin{aligned}
v^2 &= u^2 + 2as \\
&= 0 + 2(9.8)(1.5) \\
&= 29.4
\end{aligned}
$$

Let e be the maximum extension of the string.
Energy balance:

$$
\begin{aligned}
E_{ki} + E_{gp} &= E_{ep} \\
\tfrac{1}{2}mv^2 + mg\Delta h &= \tfrac{1}{2}kx^2 \\
\tfrac{1}{2}(0.6)(29.4 + (0.6)(9.8)e &= \tfrac{1}{2}(9.8)e^2 \\
8.82 + 5.88e &= 4.9e^2 \\
4.9e^2 - 5.88e - 8.82 &= 0 \\
\Rightarrow e &= 2.07 \ (-0.87 \text{ is extraneous}) \\
\Rightarrow OA &= 2.07 + 1.5 = 3.57\,\text{m}
\end{aligned}
$$

(b) Force balance:

$$
\begin{aligned}
T - W &= ma \\
kx - mg &= ma \\
(9.8)(2.07) - (0.6)(9.8) &= 0.6a \\
\therefore a &= 24.0\,\text{m s}^{-2}
\end{aligned}
$$

4. Define left to be the positive x direction.

   (a)

$$\begin{aligned}
\overline{x} &= \frac{\sum xm}{\sum m} \\
&= \frac{\frac{1}{4}kh\frac{1}{3}\pi r^2 kh - \frac{1}{4}2h\frac{1}{3}\pi r^2 2h}{\frac{1}{3}\pi r^2(kh+2h)} \\
&= \frac{\frac{1}{4}k^2h^2 - h^2}{kh+2h} \\
&= \frac{\frac{1}{4}h(k^2-4)}{k+2} \\
&= \tfrac{1}{4}h(k-2), \text{ QED}
\end{aligned}$$

   (b)

$$\begin{aligned}
\tan\theta &= \frac{\frac{1}{4}h(k-2)}{r} \\
&= \frac{\frac{1}{4}h(6-2)}{3h} \\
&= \frac{1}{3} \\
\therefore \theta &= 18.4^\circ
\end{aligned}$$

5. (a) Vertical force balance:

$$\begin{aligned}
T_A\cos 30^\circ - T_B\cos 30^\circ - W &= 0 \\
\frac{\sqrt{3}}{2}(T_A - T_B) - mg &= 0 \\
T_A - T_B &= \frac{2\sqrt{3}}{3}mg \ldots ①
\end{aligned}$$

   Horizontal force balance:

$$\begin{aligned}
&[r = \tfrac{1}{2}l\tan 30^\circ = \tfrac{\sqrt{3}}{6}l] \\
&T_A\cos 60^\circ + T_B\cos 60^\circ = mr\omega^2 \\
&\tfrac{1}{2}(T_A + T_B) = m\tfrac{\sqrt{3}}{6}l\omega^2 \\
&T_A + T_B = \tfrac{\sqrt{3}}{3}ml\omega^2 \ldots ②
\end{aligned}$$

   Solving:

$$\begin{aligned}
① + ② : 2T_A &= \tfrac{2\sqrt{3}}{3}mg + \tfrac{\sqrt{3}}{3}ml\omega^2 \\
T_A &= \tfrac{\sqrt{3}}{6}m(2g + l\omega^2), \text{ QED}
\end{aligned}$$

$$\begin{aligned}
② - ① : 2T_B &= \tfrac{\sqrt{3}}{3}ml\omega^2 - \tfrac{2\sqrt{3}}{3}mg \\
T_B &= \tfrac{\sqrt{3}}{6}m(l\omega^2 - 2g)
\end{aligned}$$

(b)

$$\begin{aligned}
T_B &> 0 \\
\therefore 0 &< \tfrac{\sqrt{3}}{6} m(l\omega^2 - 2g) \\
0 &< l\omega^2 - 2g \\
2g &< l\omega^2 \\
\omega^2 &> \frac{2g}{l} \\
\omega &> \sqrt{\frac{2g}{l}} \\
\frac{2\pi}{T} &> \sqrt{\frac{2g}{l}} \\
\frac{T}{2\pi} &< \sqrt{\frac{l}{2g}} \\
T &< \pi\sqrt{\frac{4l}{2g}} \\
&< \pi\sqrt{\frac{2l}{g}}, \text{ QED}
\end{aligned}$$

6. (a) Energy balance:

$$\begin{aligned}
E_{ki} &= E_{kf} + E_{gp} \\
\tfrac{1}{2}mv_i^2 &= \tfrac{1}{2}mv_f^2 + mg\Delta h \\
\frac{7gl}{2} &= v_f^2 + 2gl \\
v_f^2 &= \frac{3gl}{2} \\
\Rightarrow v_f &= \sqrt{\frac{3gl}{2}}
\end{aligned}$$

(b) Energy balance at top (r = radius of second circle):

$$\begin{aligned}
E_{ki} &= E_{kf} + E_{gp} \\
\tfrac{1}{2}mv_i^2 &= \tfrac{1}{2}mv_f^2 + mg\Delta h \\
v_f^2 &= v_i^2 - 2g\Delta h \\
&= \frac{3gl}{2} - 2gr
\end{aligned}$$

Force balance at top:

$$\begin{aligned}
T + W &= \frac{mv^2}{r} \\
T + 2mg &= \frac{2m}{r}\left(\frac{3gl}{2} - 2gr\right) \\
T &= \frac{3mgl}{r} - 6mg
\end{aligned}$$

$$
\begin{aligned}
0 &\leqslant \frac{3mgl}{r} - 6mg \\
\frac{3mgl}{r} &\geqslant 6mg \\
\frac{l}{r} &\geqslant 2 \\
\therefore r &\leqslant \tfrac{1}{2}l \\
\Rightarrow AB &\geqslant \tfrac{1}{2}l \quad (\because\ AB = l - r),\ \text{QED}
\end{aligned}
$$

7. $k = \frac{\lambda}{\ell} = \frac{15}{1.2} = 12.5\,\text{N m}^{-1}$

(a)

$$
\begin{aligned}
T &= -kx \\
ma &= -kx \\
a &= -\frac{k}{m}x \ \Rightarrow \text{SHM} \\
\Rightarrow \omega^2 &= \frac{k}{m} \\
&= \frac{12.5}{0.5} \\
&= 25 \\
\therefore \omega &= 5 \\
\Rightarrow T &= \frac{2\pi}{5}\,\text{s}
\end{aligned}
$$

(b)

$$
\begin{aligned}
v_{\text{max}} &= \omega A \\
&= (5)(0.8) \\
&= 4\,\text{m s}^{-1}
\end{aligned}
$$

(c)

$$
\begin{aligned}
x &= 0.8\cos 5t \\
-0.6 &= 0.8\cos 5t \\
\cos 5t &= -\tfrac{3}{4} \\
t &= 0.484\,\text{s}
\end{aligned}
$$

(d)

$$
\begin{aligned}
T &= -kx \\
ma &= -kx \\
a &= -\frac{k}{m}x \ \Rightarrow \text{SHM}
\end{aligned}
$$

(e) Momentum balance:

$$
\begin{aligned}
mv_i &= mv_f \\
(0.5)(4) &= 0.8v_f \\
v_f &= 2.5\,\text{m s}^{-1} \\
\therefore \omega^2 &= \frac{12.5}{0.8} = 15.625 \\
\Rightarrow \omega &= 3.95 \\
v_{\text{max}} &= \omega A \\
2.5 &= 3.95A \\
\Rightarrow A &= 0.632\,\text{m}
\end{aligned}
$$