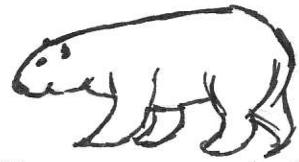
FP1 S16 UK

1. Given that k is a real number and that

$$\mathbf{A} = \begin{pmatrix} 1+k & k \\ k & 1-k \end{pmatrix}$$



find the exact values of k for which $\mathbf{A}$ is a singular matrix. Give your answers in their simplest form.

(3)

If $\text{Det}(A) = 0 \Rightarrow$ Singular

$\text{Det}(A) = (1+k)(1-k) - k^2 = 1 - 2k^2$

$\therefore$ If singular $2k^2 = 1 \quad \therefore k^2 = \frac{1}{2} \quad \therefore k = \pm\sqrt{\frac{1}{2}}$

$k = \pm \frac{\sqrt{2}}{2}$

2.

$$f(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125, \qquad x > 0$$

(a) Find $f'(x)$.

(2)

The equation $f(x) = 0$ has a root α in the interval $[12, 13]$.

(b) Using $x_0 = 12.5$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(4)

a) $f'(x) = \frac{9}{2}x^{\frac{1}{2}} + \frac{25}{2}x^{-\frac{3}{2}}$

b) $x_0 = 12.5 \Rightarrow x_1 = 12.5 - \dfrac{f(12.5)}{f'(12.5)}$

$x_1 = 12.5 - \dfrac{3(12.5)^{\frac{3}{2}} - 25(12.5)^{-\frac{1}{2}} - 125}{\frac{9}{2}(12.5)^{\frac{1}{2}} + \frac{25}{2}(12.5)^{-\frac{3}{2}}}$

$\therefore x_1 = 12.468$

3. (a) Using the formula for $\sum_{r=1}^{n} r^2$ write down, in terms of n only, an expression for

$$\sum_{r=1}^{3n} r^2$$

(1)

(b) Show that, for all integers n, where $n > 0$

$$\sum_{r=2n+1}^{3n} r^2 = \frac{n}{6}(an^2 + bn + c)$$

where the values of the constants a, b and c are to be found.

(4)

a) $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\Rightarrow \sum_{r=1}^{3n} r^2 = \frac{1}{6}(3n)(3n+1)(6n+1)$

b) $\sum_{r=2n+1}^{3n} r^2 = \sum_{r=1}^{3n} r^2 - \sum_{r=1}^{2n} r^2$

$= \frac{1}{6}(3n)(3n+1)(6n+1) - \frac{1}{6}(2n)(2n+1)(4n+1)$

$= \frac{1}{6}n\left[(9n+3)(6n+1) - (4n+2)(4n+1)\right]$

$= \frac{1}{6}n\left[54n^2 + 27n + 3 - 16n^2 - 12n - 2\right]$

$= \frac{1}{6}n\left[38n^2 + 15n + 1\right]$ $\quad a = 38 \quad b = 15 \quad c = 1$

4.

$$z = \frac{4}{1+\mathrm{i}}$$

Find, in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$

(a) z

(2)

(b) z^2

(2)

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

(c) find the value of p and the value of q.

(3)

a) $Z = \frac{4(1-i)}{(1+i)(1-i)} = \frac{4-4i}{1-i^2} = \frac{4-4i}{2} = 2-2i$

b) $z^2 = (2-2i)^2 = 4-8i+4i^2 = -8i$

c) roots α, β $\qquad \alpha = 2-2i \Rightarrow \beta = 2+2i$

$x^2 - (\alpha+\beta)x + (\alpha\beta)$

$x^2 - 4x + (4-4i^2) \Rightarrow x^2 - 4x + 8$ $\qquad p = -4$, $q = 8$

5. Points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p^2 \neq q^2$, lie on the parabola $y^2 = 4ax$.

(a) Show that the chord PQ has equation

$$y(p+q) = 2x + 2apq$$

(5)

Given that this chord passes through the focus of the parabola,

(b) show that $pq = -1$

(1)

(c) Using calculus find the gradient of the tangent to the parabola at P.

(2)

(d) Show that the tangent to the parabola at P and the tangent to the parabola at Q are perpendicular.

(2)

a) $M_{PQ} = \dfrac{2aq - 2ap}{aq^2 - ap^2} = \dfrac{2\cancel{a}\cancel{(q-p)}}{\cancel{a}\cancel{(q-p)}(q+p)} = \dfrac{2}{q+p}$

$y - y_1 = m(x - x_1) \Rightarrow y - 2ap = \dfrac{2}{q+p}(x - ap^2)$

$\Rightarrow y(q+p) - 2ap(q+p) = 2x - 2ap^2$

$\Rightarrow y(q+p) = 2x - \cancel{2ap^2} + 2apq + \cancel{2ap^2}$

$\therefore y(p+q) = 2x + 2apq$ #

b) chord passes through focus $(a, 0)$

$\Rightarrow 0(p+q) = 2a + 2apq$

$\Rightarrow -2a = 2apq \quad \therefore pq = \dfrac{-2a}{2a} \quad \therefore pq = -1$ #

c) $y^2 = 4ax \Rightarrow \frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{4a}{2y}$

at P $y = 2ap$ $\therefore M_t = \frac{4a}{2(2ap)} = \frac{4\cancel{a}}{\cancel{4a}p} = \frac{1}{p}.$

d)

If perp $M_{t_p} \times M_{t_q} = -1$

$M_{t_p} = \frac{1}{p} \quad M_{t_q} = \frac{1}{q}$

$M_{t_p} \times M_{t_q} = \frac{1}{p} + \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1$

#

6.

$$\mathbf{P} = \begin{pmatrix} -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **P**. **(2)**

The transformation U maps the point A, with coordinates (p, q), onto the point B, with coordinates $(6\sqrt{2}, 3\sqrt{2})$.

(b) Find the value of p and the value of q. **(3)**

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$.

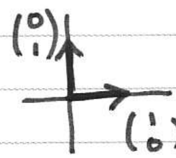
(c) Write down the matrix **Q**. **(1)**

The transformation U followed by the transformation V is the transformation T. The transformation T is represented by the matrix **R**.

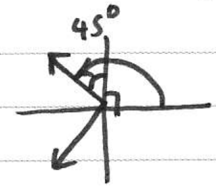
(d) Find the matrix **R**. **(3)**

(e) Deduce that the transformation T is self-inverse. **(1)**

a) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$



$\Rightarrow \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$



rotation about (0,0) through 135°.

b) $PA = B \Rightarrow A = P^{-1}(B)$

$P^{-1} = \dfrac{1}{\det(P)} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ $\quad \det(P) = \frac{1}{2} - (-\frac{1}{2}) = 1$

$\therefore P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6\sqrt{2} \\ 3\sqrt{2} \end{pmatrix} = \begin{pmatrix} -6+3 \\ -6-3 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection through $y=x$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

y becomes x, x becomes y

d)

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = R$$

e) $\det(R) = \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} - \frac{1}{2} = -1$

$$R^{-1} = -1 \begin{pmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore R = R^{-1}$ $\therefore$ Self Inverse

2

7. A complex number z is given by

$$z = a + 2\text{i}$$

where a is a non-zero real number.

(a) Find $z^2 + 2z$ in the form $x + \text{i}y$ where x and y are real expressions in terms of a. **(4)**

Given that $z^2 + 2z$ is real,

(b) find the value of a. **(1)**

Using this value for a,

(c) find the values of the modulus and argument of z, giving the argument in radians, and giving your answers to 3 significant figures. **(3)**

(d) Show the points P, Q and R, representing the complex numbers z, z^2 and $z^2 + 2z$ respectively, on a single Argand diagram with origin O. **(3)**

(e) Describe fully the geometrical relationship between the line segments OP and QR. **(2)**

a) $z^2 = (a+2i)^2 = a^2 + 4ai + 4i^2 = (a^2-4)+4ai$

$z^2+2z = (a^2-4)+4ai+2a+4i$

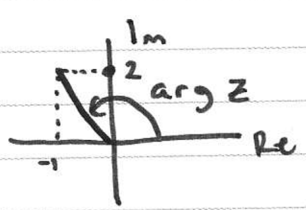
$= (a^2+2a-4)+(4a+4)i$ $\qquad x = a^2+2a-4$

$\qquad y = 4a+4.$

b) z^2+2z is Real $\Rightarrow \text{Im} = 0$ $\quad \therefore 4a+4=0$

$a = -1$

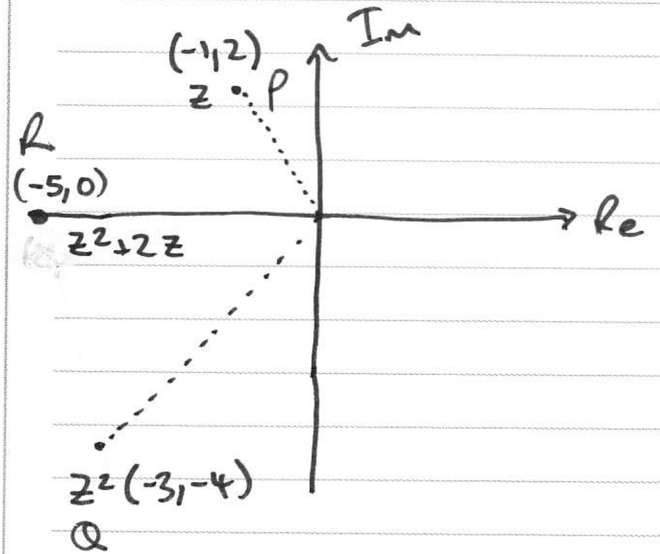
c) $|z| = |-1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$



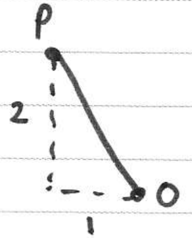
$\arg z = \pi - \tan^{-1}\left(\frac{2}{1}\right)$

$= 2.03^c$

d) $z = -1+2i$ $\quad z^2 = -3-4i$ $\quad z^2+2z = -5$

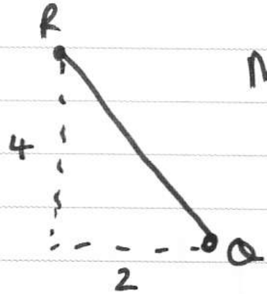


e) $\overrightarrow{OP}$



$M_{\overrightarrow{OP}} = -2 \qquad |\overrightarrow{OP}| = \sqrt{2^2+1^2} = \sqrt{5}$

$\overrightarrow{QR}$



$M_{\overrightarrow{QR}} = -2 \qquad |\overrightarrow{QR}| = \sqrt{4^2+2^2} = \sqrt{20}$

$= 2\sqrt{5}$

$\overrightarrow{OP}$ and $\overrightarrow{QR}$ are parallel

$\overrightarrow{QR}$ is twice the length of $\overrightarrow{OP}$

$\overrightarrow{QR} = 2 \times \overrightarrow{OP}$

8. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

(5)

(ii) A sequence of positive rational numbers is defined by

$$u_1 = 3$$

$$u_{n+1} = \frac{1}{3}u_n + \frac{8}{9}, \qquad n \in \mathbb{Z}^+$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 5 \times \left(\frac{1}{3}\right)^n + \frac{4}{3}$$

(5)

a) $n=1$ $\quad \sum_{1}^{1} \frac{2r+1}{r^2(r+1)^2} = \frac{3}{1(2)^2} = \frac{3}{4}$

$1 - \frac{1}{(1+1)^2} = 1 - \frac{1}{4} = \frac{3}{4}$ $\quad \therefore$ LHS = RHS true for $n=1$

assume true for $n=k$.

$$\sum_{1}^{k} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+1)^2}$$

$n=k+1$ $\quad \sum_{1}^{k+1} \frac{2r+1}{r^2(r+1)^2} = \sum_{1}^{k} \frac{2r+1}{r^2(r+1)^2} + \frac{2(k+1)+1}{(k+1)^2(k+2)^2}$

$$= \left(1 - \frac{1}{(k+1)^2}\right) + \frac{2k+3}{(k+1)^2(k+2)^2}$$

$$= 1 - \left(\frac{1}{(k+1)^2}\frac{(k+2)^2}{(k+2)^2} - \frac{2k+3}{(k+1)^2(k+2)^2}\right)$$

$$= 1 - \frac{k^2+4k+4-2k-3}{(k+1)^2(k+2)^2} = 1 - \frac{k^2+2k+1}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{(k+1)^2}{(k+1)^2(k+2)^2} = 1 - \frac{1}{(k+2)^2}$$

$n=k+1$ RHS $= 1 - \frac{1}{(k+1+1)^2} = 1 - \frac{1}{(k+2)^2}$

$\therefore$ LHS = RHS $\therefore$ true for $n=k+1$

true for $n=1$, if true for $n=k$ then true for $n=k+1$
$\therefore$ by Mathematical Induction true for all $n \in \mathbb{Z}^+$

(ii) $u_1 = 3$ $\quad u_1 = 5\left(\frac{1}{3}\right)^1 + \frac{4}{3} = \frac{5}{3} + \frac{4}{3} = 3$

LHS = RHS $\therefore$ true for $n=1$

$u_2 = \frac{1}{3}(3) + \frac{8}{9} = \frac{17}{9}$

$u_2 = 5\left(\frac{1}{3}\right)^2 + \frac{4}{3} = \frac{5}{9} + \frac{12}{9} = \frac{17}{9}$

LHS = RHS $\therefore$ true for $n=2$

assume true for $n=k \Rightarrow u_k = 5\left(\frac{1}{3}\right)^k + \frac{4}{3}$

$n=k+1$ $\quad u_{k+1} = \frac{1}{3}u_k + \frac{8}{9} = \frac{5}{3}\left(\frac{1}{3}\right)^k + \frac{4}{9} + \frac{8}{9}$

$$= \frac{5}{3}\left(\frac{1}{3}\right)^k + \frac{12}{9} = 5\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^k + \frac{4}{3}$$

$$= 5\left(\frac{1}{3}\right)^{k+1} + \frac{4}{3}$$

RHS $\quad u_{k+1} = 5\left(\frac{1}{3}\right)^{k+1} + \frac{4}{3}$

$\therefore$ RHS = LHS $\therefore$ true for $n = k+1$

true for $n=1, n=2$ true for $n=k+1$ if true for $n=k$ $\therefore$ by Mathematical Induction true for all $n \in \mathbb{Z}^+$.

9. The rectangular hyperbola, H, has cartesian equation $xy = 25$

(a) Show that an equation of the normal to H at the point $P\left(5p, \frac{5}{p}\right)$, $p \neq 0$, is

$$y - p^2x = \frac{5}{p} - 5p^3$$

(5)

This normal meets the line with equation $y = -x$ at the point A.

(b) Show that the coordinates of A are

$$\left(-\frac{5}{p} + 5p, \frac{5}{p} - 5p\right)$$

(3)

The point M is the midpoint of the line segment AP.
Given that M lies on the positive x-axis,

(c) find the exact value of the x coordinate of point M.

(3)

a) $y = 25x^{-1} \Rightarrow \frac{dy}{dx} = -25x^{-2} = \frac{-25}{x^2}$

$\Rightarrow$ at P $\quad M_t = \frac{-25}{25p^2} = \frac{-1}{p^2}$

$M_n = p^2$

$y - y_1 = m(x - x_1)$

$y - \frac{5}{p} = p^2(x - 5p) \Rightarrow y - \frac{5}{p} = p^2x - 5p^3$

$\therefore y - p^2x = \frac{5}{p} - 5p^3$

b) $y = -x \Rightarrow -x - p^2x = \frac{5}{p} - 5p^3$

$\Rightarrow x + p^2x = 5p^3 - \frac{5}{p} = \frac{5p^4 - 5}{p}$

$\Rightarrow x(1 + p^2) = \frac{5(p^4 - 1)}{p}$

$\Rightarrow x = \frac{5\cancel{(p^2+1)}(p^2-1)}{p\cancel{(p^2+1)}} = \frac{5p^2 - 5}{p}$

$\therefore x = 5p - \frac{5}{p}$ $\quad y = \frac{5}{p} - 5p$ $\quad \left(5p - \frac{5}{p}, \frac{5}{p} - 5p\right)$

#

c) Midpoint AP

$$= \left(\frac{5p - \frac{5}{p} + 5p}{2} , \frac{\frac{5}{p} + \frac{5}{p} - 5p}{2} \right)$$

$$= \left(5p - \frac{5}{2p} , \frac{5}{p} - \frac{5p}{2} \right)$$

M lies on x-axis $\therefore y = 0$

$$\frac{5}{p} = \frac{5p}{2} \Rightarrow \quad 10 = 5p^2 \quad \therefore p^2 = 2$$

$$\therefore p = \sqrt{2}$$

(not $-\sqrt{2}$ as it is on the x-axis)

$$x = 5\sqrt{2} - \frac{5}{2\sqrt{2}} = 5\sqrt{2} - \frac{5\sqrt{2}}{4} = \frac{15\sqrt{2}}{4}$$