1. (a) Express $3\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Give the exact value of R and give the value of α to 2 decimal places.

(3)

(b) Hence solve, for $0 \leqslant \theta < 360°$, the equation

$$3\cos\theta + 5\sin\theta = 2$$

Give your answers to one decimal place.

(4)

(c) Use your solutions to parts (a) and (b) to deduce the smallest positive value of θ for which

$$3\cos\theta - 5\sin\theta = 2$$

(2)

a) $3\cos\theta + 5\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R = \sqrt{3^2 + 5^2} = \sqrt{34}$, $\tan\alpha = \frac{5}{3}$, $\alpha = 59.04°$

b) $\sqrt{34}\cos(\theta - 59.04°) = 2$

$\cos(\theta - 59.04) = \frac{2}{\sqrt{34}}$

$\theta - 59.04 = 69.94$, $\theta - 59.04 = 290.06$

$\theta = 128.98°$ $\qquad$ $\theta = 349.1°$

$\therefore \theta = 129.0°, 349.1°$

c) $\cos(\theta + 59.04°) = \frac{2}{\sqrt{34}}$

$\theta \longrightarrow \theta + 2 \times 59.04°$

$\theta + 118.08 = 128.98$

$\theta = 10.9°$

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2. The point P with coordinates $\left(\frac{\pi}{2}, 1\right)$ lies on the curve with equation

$$4x\sin x = \pi y^2 + 2x, \qquad \frac{\pi}{6} \leqslant x \leqslant \frac{5\pi}{6}$$

Find an equation of the normal to the curve at P.

(6)

$4\sin x + 4x\cos x = 2y\pi \frac{dy}{dx} + 2$

$x = \frac{\pi}{2}, \quad y = 1$

$4(1) + 4(0) = 2\pi \frac{dy}{dx} + 2$

$2\pi \frac{dy}{dx} = 2 \longrightarrow \frac{dy}{dx} = \frac{1}{\pi}$

$\therefore m = -\pi$

$\frac{y-1}{x - \frac{\pi}{2}} = -\pi$

$y - 1 = -\pi x + \frac{\pi^2}{2}$

$y = -\pi x + \frac{\pi^2}{2} + 1$

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3. (a) Find the binomial expansion of

$$(1 + ax)^{-3}, \quad |ax| < 1$$

in ascending powers of x, up to and including the term in x^3, giving each coefficient as simply as possible in terms of the constant a.

(3)

$$f(x) = \frac{2 + 3x}{(1 + ax)^3}, \quad |ax| < 1$$

In the series expansion of $f(x)$, the coefficient of x^2 is 3

Given that $a < 0$

(b) find the value of the constant a,

(4)

(c) find the coefficient of x^3 in the series expansion of $f(x)$, giving your answer as a simplified fraction.

(2)

a)

$$(1+ax)^{-3} = 1 - 3ax + \frac{-3 \times -4}{2}(ax)^2 + \frac{-3\times-4\times-5}{6}(ax)^3$$

$$= 1 - 3ax + 6a^2x^2 - 10a^3x^3 + \cdots$$

b) $(2+3x)(1-3ax+6a^2x^2-10a^3x^3)$

Coeffcient of x^2: $12a^2 - 9a$

$$\therefore \quad 12a^2 - 9a - 3 = 0$$

$$4a^2 - 3a - 1 = 0$$

$$(a-1)(4a+1) = 0$$

$a = 1$ rej. $\quad \boxed{a = -\frac{1}{4}}$

c) coeffcient of x^3: $-20a^3 + 18a^2$

$$= \boxed{\frac{23}{16}}$$

blank

4. $$g(x) = \frac{x^4 + x^3 - 7x^2 + 8x - 48}{x^2 + x - 12}, \qquad x > 3, \quad x \in \mathbb{R}$$

(a) Given that

$$\frac{x^4 + x^3 - 7x^2 + 8x - 48}{x^2 + x - 12} \equiv x^2 + A + \frac{B}{x - 3}$$

find the values of the constants A and B.

(4)

(b) Hence, or otherwise, find the equation of the tangent to the curve with equation $y = g(x)$ at the point where $x = 4$. Give your answer in the form $y = mx + c$, where m and c are constants to be determined.

(5)

a)

$$x^2 + 5$$

$$x^2 + x - 12 \ \big) \ x^4 + x^3 - 7x^2 + 8x - 48$$

$$\ominus \quad x^4 + x^3 - 12x^2$$

$$5x^2 + 8x - 48$$

$$\ominus \quad 5x^2 + 5x - 60$$

$$3x + 12$$

$$\therefore \ g(x) = x^2 + 5 + \frac{3x + 12}{(x-3)(x+4)}$$

$$= x^2 + 5 + \frac{3\cancel{(x+4)}}{(x-3)\cancel{(x+4)}} = x^2 + 5 + \frac{3}{x-3}$$

b) $$\frac{dy}{dx} = 2x - \frac{3}{(x-3)^2}$$

at $x = 4 \longrightarrow \dfrac{dy}{dx} = 5, \ y = 24$

$$\frac{y - 24}{x - 4} = 5, \quad y - 24 = 5x - 20$$

$$y = 5x + 4$$

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5. Use integration by parts to find the exact value of

$$\int_0^2 x2^x \, dx$$

Write your answer as a single simplified fraction.

(6)

$u = x \qquad dv = 2^x \, dx$

$du = dx \qquad v = \dfrac{2^x}{\ln 2}$

$$= \left[\frac{x\,2^x}{\ln 2}\right]_0^2 - \int_0^2 \frac{2^x}{\ln 2}\,dx$$

$$= \frac{8}{\ln 2} - 0 - \frac{1}{(\ln 2)^2}\left[2^x\right]_0^2$$

$$= \frac{8}{\ln 2} - \frac{1}{(\ln 2)^2}\left[4 - 1\right]$$

$$= \frac{8}{\ln 2} - \frac{3}{(\ln 2)^2}$$

$$= \frac{8\ln 2 - 3}{(\ln 2)^2}$$

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6. Given that a and b are constants and that $a > b > 0$

(a) on separate diagrams, sketch the graph with equation

(i) $y = |x - a|$

(ii) $y = |x - a| - b$

Show on each sketch the coordinates of each point at which the graph crosses or meets the x-axis and the y-axis.

(5)

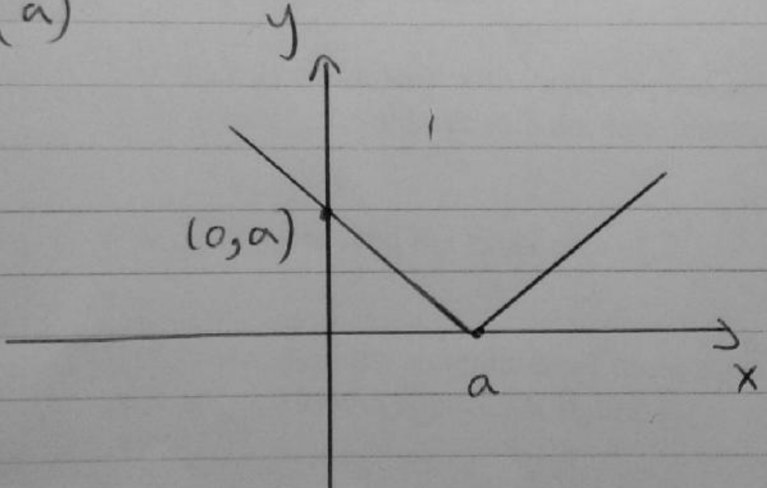
(b) Hence or otherwise find the complete set of values of x for which

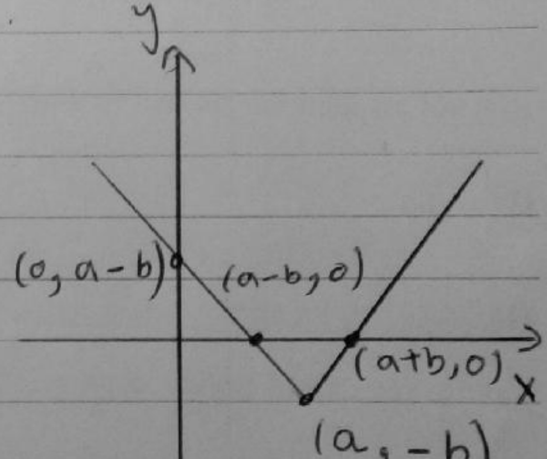
$$|x - a| - b < \frac{1}{2}x$$

giving your answer in terms of a and b.

(4)

(a)





$x - a - b = 0,\ x = a + b$ | $-x + a - b = 0$

$x = a - b$

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Question 6 continued

b) $x - a - b < \frac{1}{2}x$

$\frac{1}{2}x < a + b \rightarrow x < 2a + 2b$

$-x + a - b < \frac{1}{2}x$

$\frac{3}{2}x > a - b \longrightarrow x > \frac{2}{3}(a-b)$

$\therefore \frac{2}{3}(a-b) < x < 2(a+b)$

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7.

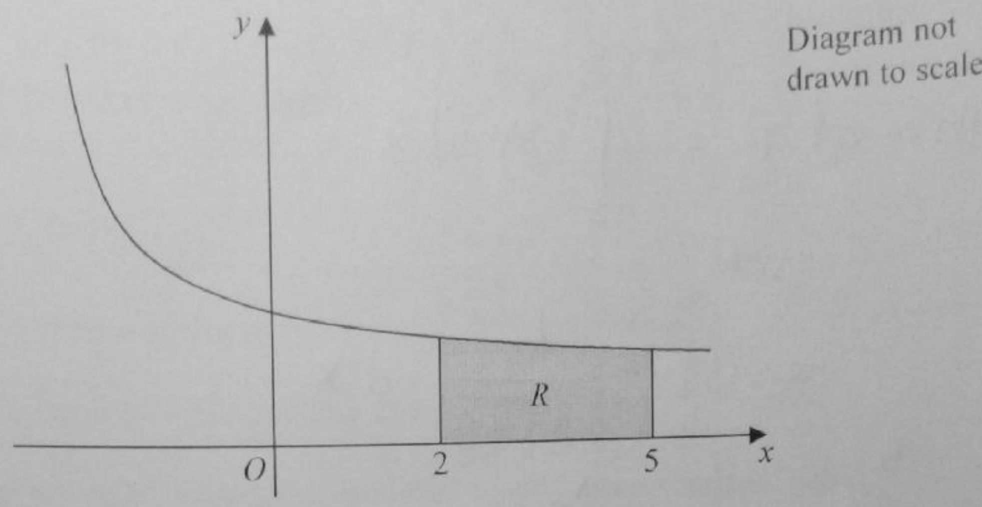

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \frac{1}{\sqrt{2x+5}}$, $x > -2.5$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the lines with equations $x = 2$ and $x = 5$

(a) Use the trapezium rule with three strips of equal width to find an estimate for the area of R, giving your answer to 3 decimal places.

(4)

(b) Use calculus to find the exact area of R.

(4)

(c) Hence calculate the magnitude of the error of the estimate found in part (a), giving your answer to one significant figure.

(1)

x	2	3	4	5
y	1/3	0.3015	0.27735	0.2582

$h = \frac{5-2}{3} = 1$

area $= \frac{1}{2}(1)\left[\frac{1}{3} + 0.2582 + 2(0.3015 + 0.27735)\right]$

$= 0.875$

b) Area of R $= \int_2^5 y \, dx$

$$= \int_2^5 \frac{1}{\sqrt{2x+5}} \, dx$$

$$= \int_2^5 (2x+5)^{-\frac{1}{2}} \, dx = \frac{2}{2}\left[\sqrt{2x+5}\right]_2^5$$

$$= \sqrt{15} - 3$$

(c) absolute error $= 0.875 - (\sqrt{15} - 3)$

$$= 0.002$$

Leave blank

8. (a) Prove that

$$\sin 2x - \tan x \equiv \tan x \cos 2x, \qquad x \neq \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $0 \leqslant \theta < \frac{\pi}{2}$

(i) $\sin 2\theta - \tan \theta = \sqrt{3} \cos 2\theta$

(ii) $\tan(\theta + 1)\cos(2\theta + 2) - \sin(2\theta + 2) = 2$

Give your answers in radians to 3 significant figures, as appropriate.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(7)

(a) $2\sin x \cos x - \frac{\sin x}{\cos x} = \frac{2\sin x \cos^2 x - \sin x}{\cos x}$

$= \frac{\sin x}{\cos x}(2\cos^2 x - 1) = \tan x \cos 2x$

$= R.H.S$

b) (i) $\tan\theta \cos 2\theta = \sqrt{3}\cos 2\theta$

$\cos 2\theta [\tan\theta - \sqrt{3}] = 0$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, 2\theta = \frac{3}{2}\pi$

$\theta = \frac{\pi}{4}, \theta = \frac{3}{4}\pi$ ↘ rej.

$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}, \theta = \frac{4}{3}\pi$ ↘ rej.

$\theta = \frac{\pi}{4}, \frac{\pi}{3}$

ii) $\tan(\theta+1)\cos(2(\theta+1)) - \sin(2(\theta+1)) = 2$

let $\theta + 1 = x$

$\tan(x)\cos(2x) - \sin(2x) = 2$

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$\cancel{\sin 2x} - \tan x - \cancel{\sin 2x} = 2$

$\tan x = -2$

$x = -1.107 \qquad x = 2.03$

$\theta + 1 = -1.107 \quad , \quad \theta + 1 = 2.03$

$\theta = -2.107 \quad , \quad \theta = 1.03$

↳ rej.

$\therefore \quad \theta = 1.03$

9.

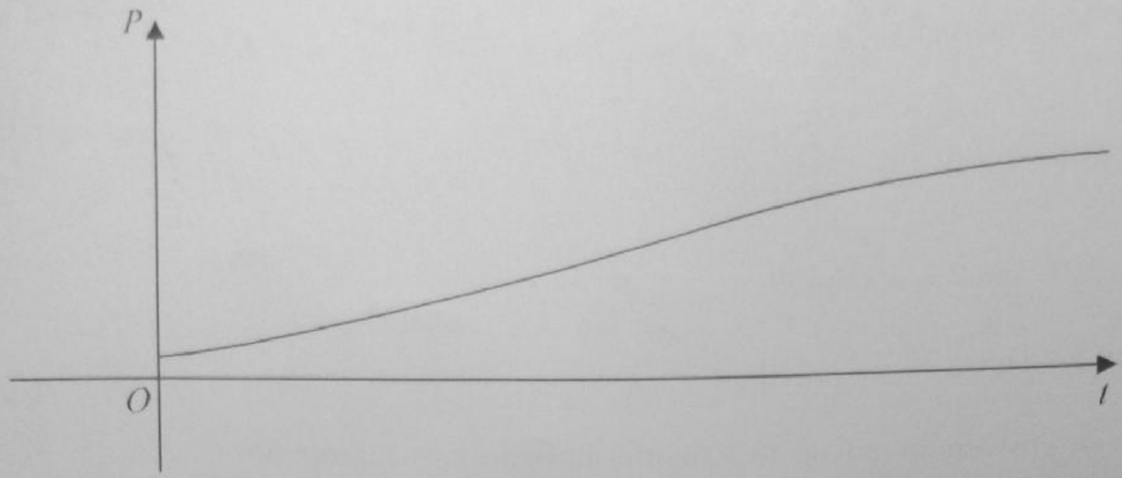


Figure 2

The population of a species of animal is being studied. The population P, at time t years from the start of the study, is assumed to be

$$P = \frac{9000e^{kt}}{3e^{kt} + 7}, \qquad t \geqslant 0$$

where k is a positive constant.

A sketch of the graph of P against t is shown in Figure 2.

Use the given equation to

(a) find the population at the start of the study,

(2)

(b) find the value for the upper limit of the population.

(1)

Given that $P = 2500$ when $t = 4$

(c) calculate the value of k, giving your answer to 3 decimal places.

(5)

Using this value for k,

(d) find, using $\frac{dP}{dt}$, the rate at which the population is increasing when $t = 10$

Give your answer to the nearest integer.

(3)

Question 9 continued

a) $t = 0 \rightarrow P = \dfrac{9000}{3+7} = 900$

b) $P = \dfrac{9000}{3+7e^{-kt}}$

$t \rightarrow \infty \,, \quad P = \dfrac{9000}{3} = 3000$

c) $\dfrac{2500}{9000} = \dfrac{e^{4k}}{3e^{4k}+7}$

$15e^{4k} + 35 = 18e^{4k}$

$3e^{4k} = 35 \,, \quad e^{4k} = \dfrac{35}{3}$

$k = \frac{1}{4}\ln\left(\frac{35}{3}\right) = 0.614$

d) $\dfrac{dP}{dt} = -\dfrac{9000}{(3+7e^{-kt})^2}(-7ke^{-kt})$

$= \dfrac{63000ke^{-kt}}{(3+7e^{-kt})^2}$

$t = 4 \rightarrow \dfrac{dP}{dt} = 9.16 \simeq 9$

10. (a) Given that $-\frac{\pi}{2} < g(x) < \frac{\pi}{2}$, sketch the graph of $y = g(x)$ where

$$g(x) = \arctan x, \quad x \in \mathbb{R}$$

(2)

(b) Find the exact value of x for which

$$3g(x+1) - \pi = 0$$

(3)

The equation $\arctan x - 4 + \frac{1}{2}x = 0$ has a positive root at $x = \alpha$ radians.

(c) Show that $5 < \alpha < 6$

(2)

The iteration formula

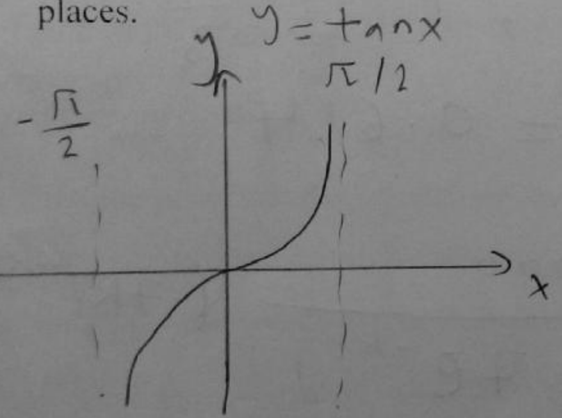
$$x_{n+1} = 8 - 2\arctan x_n$$

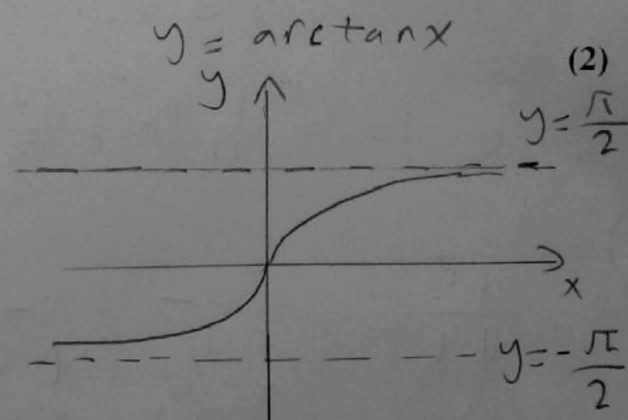
can be used to find an approximation for α

(d) Taking $x_0 = 5$, use this formula to find x_1 and x_2, giving each answer to 3 decimal places.

(2)

a)





b) $3\arctan(x+1) - \pi = 0$

$\arctan(x+1) = \frac{\pi}{3}$

$\tan\frac{\pi}{3} = x+1 \rightarrow \boxed{-x = 1 - \sqrt{3}}$

$x = 0.732 \qquad \boxed{x = \sqrt{3} - 1}$

(exact)

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Question 10 continued

c) Let $f(x) = \arctan x - 4 + \frac{1}{2}x$

$f(5) = -0.1265\ldots$, $f(6) = 0.4056\ldots$

change of sign implies there is a root between $x=5$ & $x=6$

$\therefore \quad 5 < \alpha < 6$

d) $x_1 = 8 - 2\arctan(5) = 5.253$

$x_2 = 8 - 2\arctan(x_1) = 5.235$

11. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$l_2 : \mathbf{r} = \begin{pmatrix} -6 \\ -7 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 4 \\ b \end{pmatrix}$$

where λ and μ are scalar parameters and b is a constant.

Given that l_1 and l_2 meet at the point X,

(a) show that $b = -3$ and find the coordinates of X.

(5)

The point A lies on l_1 and has coordinates (6, 3, 5)

The point B lies on l_2 and has coordinates (14, 9, – 9)

(b) Show that angle $AXB = \arccos\left(-\frac{1}{10}\right)$

(4)

(c) Using the result obtained in part (b), find the exact area of triangle AXB.

Write your answer in the form $p\sqrt{q}$ where p and q are integers to be determined.

(3)

a) $7 + \lambda = -6 + 5\mu$ | $4 + \lambda = -7 + 4\mu$

$\lambda = -13 + 5\mu$ | $-9 + 5\mu = -7 + 4\mu$

$\boxed{\lambda = -3}$ $\boxed{\mu = 2}$

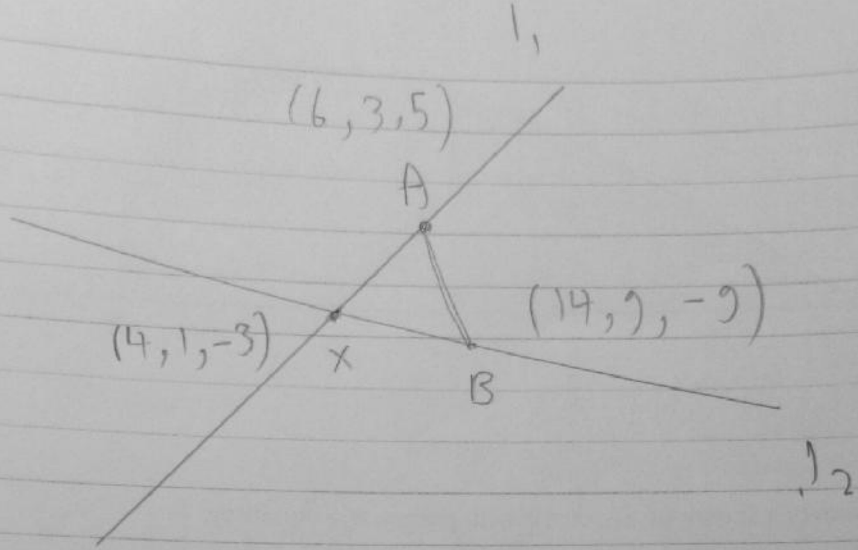
$9 + 4\lambda = 3 + b\mu$

$-3 = 3 + 2b \rightarrow 2b = -6$

$\boxed{b = -3}$

$X = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$

Question 11 continued



$$\vec{AX} = \begin{pmatrix}4\\1\\-3\end{pmatrix} - \begin{pmatrix}6\\3\\5\end{pmatrix} = \begin{pmatrix}-2\\-2\\-8\end{pmatrix}$$

$$\vec{BX} = \begin{pmatrix}4\\1\\-3\end{pmatrix} - \begin{pmatrix}14\\9\\-9\end{pmatrix} = \begin{pmatrix}-10\\-8\\6\end{pmatrix}$$

$$\cos\theta = \frac{a\cdot b}{|a|\,|b|}, \quad a\cdot b = -12$$

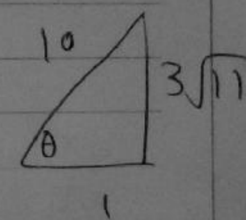
$$a = \sqrt{2^2+2^2+8^2} = 6\sqrt{2}, \quad b = 10\sqrt{2}$$

$$\cos AXB = \frac{-12}{6\sqrt{2}\times 10\sqrt{2}} = -\frac{1}{10}, \therefore AXB = \arccos\left(-\frac{1}{10}\right)$$

c) Area of $\Delta = \frac{1}{2} \times |AX|\,|BX| \sin AXB$

$$= \frac{1}{2}(6\sqrt{2})(10\sqrt{2}) \sin AXB$$

Since θ is obtuse, $\sin\theta$ is +ve



$$\therefore \sin AXB = \frac{3\sqrt{11}}{10}$$

$\therefore$ Area of $\Delta = 18\sqrt{11}$

12.

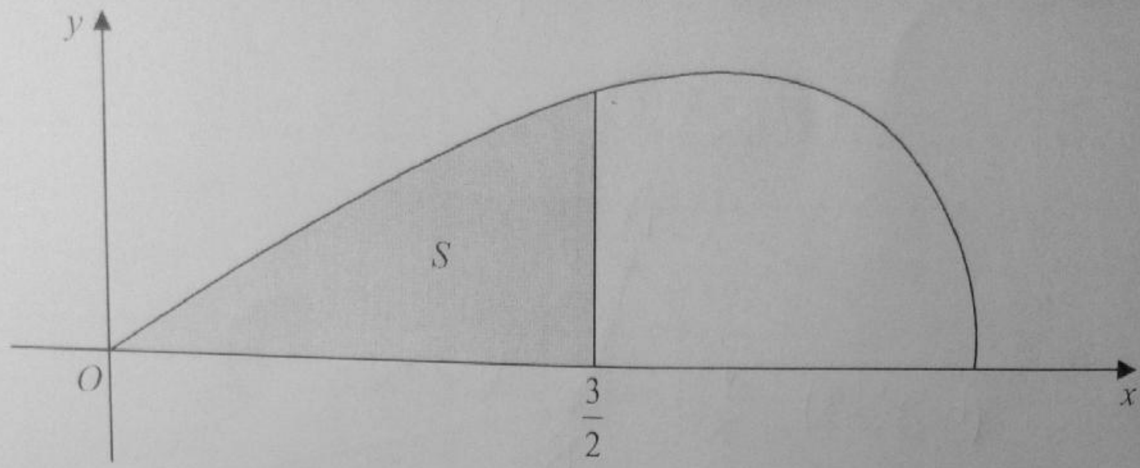


Figure 3

Figure 3 shows a sketch of the curve with parametric equations

$$x = 3\sin t, \quad y = 2\sin 2t, \quad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The finite region S, shown shaded in Figure 3, is bounded by the curve, the x-axis and the line with equation $x = \frac{3}{2}$

The shaded region S is rotated through 2π radians about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid of revolution is given by

$$k\int_0^a \sin^2 t \cos^3 t \, dt$$

where k and a are constants to be given in terms of π.

(5)

(b) Use the substitution $u = \sin t$, or otherwise, to find the exact value of this volume, giving your answer in the form $\frac{p\pi}{q}$ where p and q are integers.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

a) $V = \pi \int y^2 \frac{dx}{dt} \, dt$

$\frac{dx}{dt} = 3\cos t \qquad y^2 = 4\sin^2 2t$

$= 4(2\sin t \cos t)^2$

$= 16 \sin^2 t \cos^2 t$

Question 12 continued

a)

$$\therefore V = 48\pi \int_0^{a} \sin^2 t \cos^3 t \, dt$$

$$\frac{3}{2} = 3\sin t \longrightarrow \sin t = \frac{1}{2}$$

$$\boxed{t = \frac{\pi}{6}} \longrightarrow a = \frac{\pi}{6}$$

$$\therefore V = 48\pi \int_0^{\pi/6} \sin^2 t \cos^3 t \, dt$$

b) $u = \sin t \longrightarrow du = \cos t \, dt$

$$I = 48\pi \int_0^{1/2} u^2 \cos^2 t \, du$$

$$= 48\pi \int_0^{1/2} u^2 (1 - u^2) \, du$$

$$= 48\pi \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^{1/2}$$

$$= 48\pi \left[\frac{17}{480} \right] = \frac{17}{10}\pi$$

$$\therefore \text{Volume} = \frac{17}{10}\pi$$

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13.

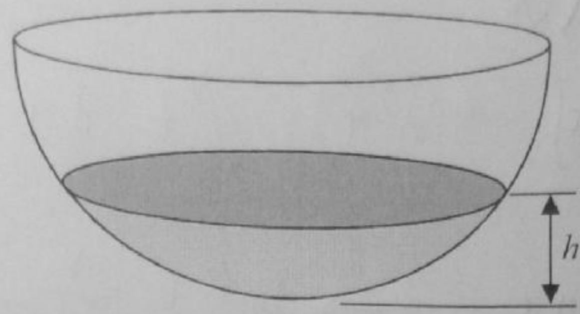

Figure 4

Figure 4 shows a hemispherical bowl containing some water.

At t seconds, the height of the water is h cm and the volume of the water is V cm^3, where

$$V = \frac{1}{3}\pi h^2(30 - h), \qquad 0 < h \leqslant 10$$

The water is leaking from a hole in the bottom of the bowl.

Given that $\frac{dV}{dt} = -\frac{1}{10}V$

(a) show that $\frac{dh}{dt} = -\frac{h(30-h)}{30(20-h)}$

(5)

(b) Write $\frac{30(20-h)}{h(30-h)}$ in partial fraction form.

(3)

Given that $h = 10$ when $t = 0$,

(c) use your answers to parts (a) and (b) to find the time taken for the height of the water to fall to 5 cm. Give your answer in seconds to 2 decimal places.

(6)

a) $\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$

$V = \frac{1}{3}\pi\,(30h^2 - h^3)$

$\frac{dv}{dh} = \frac{1}{3}\pi\left[60h - 3h^2\right)$

$= \pi\,(20h - h^2)$

$$\frac{dh}{dt} = \frac{1}{\pi(20h-h^2)} \times -\frac{1}{10} V$$

$$= \frac{-1}{10\pi(20h-h^2)} \cdot \frac{1}{3}\pi h^2 (30-h)$$

$$= -\frac{1}{30\cancel{\pi}} \cdot \frac{\cancel{\pi}\, \overset{h}{\cancel{h^2}}(30-h)}{\cancel{h}(20-h)} = -\frac{1}{30} \frac{h(30-h)}{(20-h)}$$

b) $$\frac{30(20-h)}{h(30-h)} = \frac{A}{h} + \frac{B}{30-h}$$

$$A = \frac{30(20)}{30} = 20, \quad B = \frac{3\cancel{0}(20-30)}{3\cancel{0}}$$

$$= -10$$

$$\therefore \frac{30(20-h)}{h(30-h)} = \frac{20}{h} - \frac{10}{30-h}$$

c) $$\int \frac{30(20-h)}{h(30-h)} \, dh = \int - dt$$

$$= 20\ln|h| + 10\ln|30-h| = -t + \ln K$$

$$10 \ln|h^2(30-h)| = -t + \ln K$$

$h = 10,\ t = 0 \longrightarrow \ln K = 10 \ln 2000$

$$\therefore\ t = 10 \ln \left| \frac{2000}{h^2(30-h)} \right|$$

at $h = 5 \longrightarrow t = 11.63$ sec.