C3 June 2016 Model Answers

by Kprime2 TSR

1. The functions f and g are defined by

$$f : x \to 7x - 1, \quad x \in \mathbb{R}$$

$$g : x \to \frac{4}{x-2}, \quad x \neq 2, x \in \mathbb{R}$$

(a) Solve the equation $fg(x) = x$ **(4)**

(b) Hence, or otherwise, find the largest value of a such that $g(a) = f^{-1}(a)$ **(1)**

1(a). $fg(x) = 7\left(\frac{4}{x-2}\right) - 1$

$= \frac{28}{x-2} - 1$

$fg(x) = x \Rightarrow \frac{28}{x-2} - 1 = x$

$\times\ x-2 \Rightarrow 28 - x + 2 = x(x-2)$

$\therefore x^2 - x - 30 = 0$

$(x-6)(x+5) = 0$

$\Rightarrow \underline{\underline{x = 6}} \quad \underline{\underline{x = -5}}$

(b) $fg(\alpha) = \alpha$

$\Rightarrow f^{-1}[fg(\alpha)] = f^{-1}(\alpha)$

$\therefore g(\alpha) = f^{-1}(\alpha)$

$\underline{\underline{\alpha = 6}}$

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2. $$y = \frac{4x}{x^2 + 5}$$

(a) Find $\frac{dy}{dx}$, writing your answer as a single fraction in its simplest form. **(4)**

(b) Hence find the set of values of x for which $\frac{dy}{dx} < 0$ **(3)**

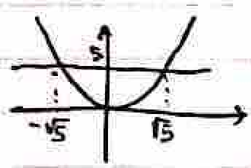
2(a) $$y = \frac{4x}{x^2+5}$$

$$\frac{\partial y}{\partial x} = \frac{4(x^2+5) - 4x(2x)}{(x^2+5)^2}$$

$$= \frac{4x^2+20 - 8x^2}{(x^2+5)^2}$$

$$\therefore \frac{\partial y}{\partial x} = \frac{20 - 4x^2}{(x^2+5)^2}$$

(b) $$20 - 4x^2 < 0$$



$$\Rightarrow x^2 > 5$$

$x > \sqrt{5}$ $\quad$ $x < -\sqrt{5}$

3. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Give the exact value of R and give the value of α to 2 decimal places.

(3)

(b) Hence solve, for $0 \leqslant \theta < 360°$,

$$\frac{2}{2\cos\theta - \sin\theta - 1} = 15$$

Give your answers to one decimal place.

(5)

(c) Use your solutions to parts (a) and (b) to deduce the smallest positive value of θ for which

$$\frac{2}{2\cos\theta + \sin\theta - 1} = 15$$

Give your answer to one decimal place.

(2)

3(a) $R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\equiv 2\cos\theta - \sin\theta$

$\Rightarrow R\cos\alpha = 2$ & $R\sin\alpha = 1$

$R^2 = 2^2+1^2 = 5 \Rightarrow R = \pm\sqrt{5}$

$\tan\alpha = \frac{1}{2} \Rightarrow \alpha = \arctan\left(\frac{1}{2}\right) = \underline{26.57°}$ (2dp)

$\therefore 2\cos\theta - \sin\theta = \underline{\sqrt{5}\cos(\theta + 26.57°)}$

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Question 3 continued

(b) $\dfrac{2}{2\cos\theta - \sin\theta - 1} = 15$

$\therefore \quad 2 = 30\cos\theta - 15\sin\theta - 15$

$17 = 15\left(2\cos\theta - \sin\theta\right)$

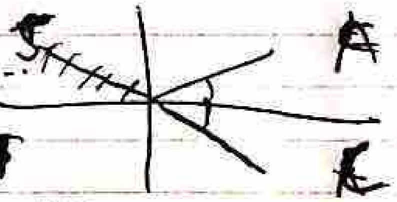
$\therefore \quad \sqrt{5}\cos(\theta + 26.57°) = \dfrac{17}{15}$

$\therefore \cos(\theta + 26.57°) = \dfrac{17}{15\sqrt{5}}$

$0 \le \theta < 360°$

$26.57 \le (\theta + 26.57°) < 386.57°$

$\theta + 26.565.. = \arccos\left(\frac{17}{15\sqrt{5}}\right) = 59.54...$



~~$\theta + 26.565... = 180 - 59.54... = 120.45...$~~

$\theta + 26.565.. = 360 - 59.54... = 300.453...$

$\therefore \quad \underline{\underline{\theta = 33.0°}} \qquad \underline{\underline{\theta = 273.9°}}$

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Question 3 continued

(c) $2\cos\theta \pm \sin\theta \equiv R\cos(\theta \mp \alpha)$

(a)

$\therefore 2\cos\theta + \sin\theta = \sqrt{5}\cos(\theta - 26.57)$

(b) from part (b):

$\cos(\theta - 26.57) = \frac{17}{15\sqrt{5}}$

$\therefore \theta - 26.565\ldots = \arccos\left(\frac{17}{15\sqrt{5}}\right)$

$= 59.546\ldots$

$\therefore \theta = 59.546\ldots + 26.565\ldots$

$\Rightarrow \theta = 86.1^\circ$ (1dp)

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4.

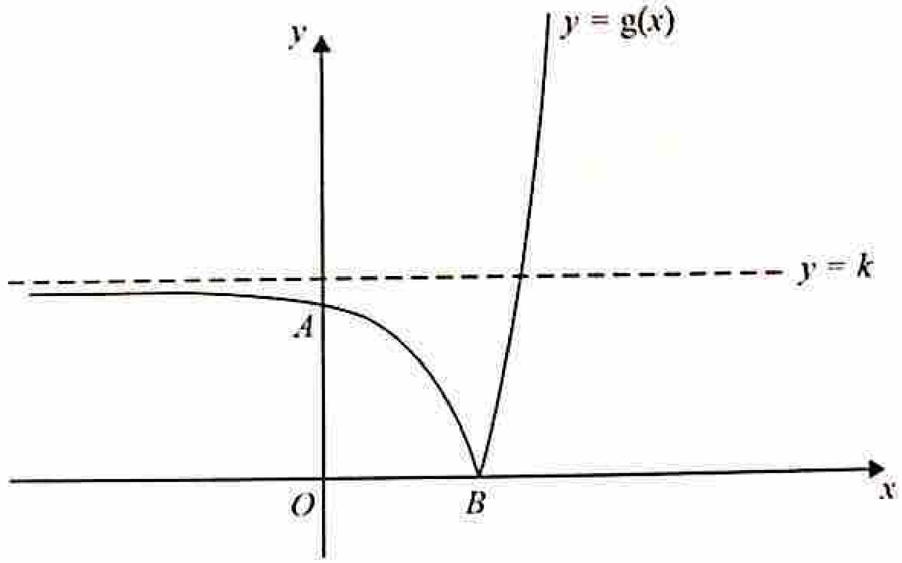


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = g(x)$, where

$$g(x) = |4e^{2x} - 25|, \qquad x \in \mathbb{R}$$

The curve cuts the y-axis at the point A and meets the x-axis at the point B. The curve has an asymptote $y = k$, where k is a constant, as shown in Figure 1

(a) Find, giving each answer in its simplest form,

(i) the y coordinate of the point A,

(ii) the exact x coordinate of the point B,

(iii) the value of the constant k.

(5)

The equation $g(x) = 2x + 43$ has a positive root at $x = \alpha$

(b) Show that α is a solution of $x = \frac{1}{2}\ln\left(\frac{1}{2}x + 17\right)$

(2)

The iteration formula

$$x_{n+1} = \frac{1}{2}\ln\left(\frac{1}{2}x_n + 17\right)$$

can be used to find an approximation for α

(c) Taking $x_0 = 1.4$ find the values of x_1 and x_2
Give each answer to 4 decimal places.

(2)

(d) By choosing a suitable interval, show that $\alpha = 1.437$ to 3 decimal places.

(2)

4(a)(i) $g(0) = -21$

$\therefore \; y_A = 21$

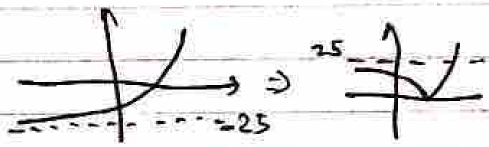
(ii) $g(x) = 0 \Rightarrow 4e^{2x} - 25 = 0$

$e^{2x} = \frac{25}{4}$

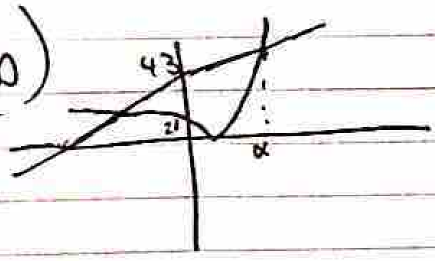
$e^{x} = \frac{5}{2}$

$x = \ln\left(\frac{5}{2}\right)$

(iii) $k = 25$



(b)



$\Rightarrow 4e^{2x} - 25 = 2x + 43$

$\therefore 4e^{2x} = 2x + 68$

$\therefore e^{2x} = \frac{2x+68}{4} = \frac{1}{2}x + 17$

$\therefore e^{x} = \sqrt{\frac{1}{2}x + 17}$

$\Rightarrow x = \ln\left(\sqrt{\frac{1}{2}x+17}\right) = \ln\left[\left(\frac{1}{2}x + 17\right)^{\frac{1}{2}}\right]$

$\therefore \; x = \frac{1}{2}\ln\left(\frac{1}{2}x + 17\right)$ as required.

Turn over

(c) $x_1 = 1.4368$ (4dp)

$x_2 = 1.4373$ (4dp)

(d) We are solving $4e^{2x} - 23 = 2x + 43$

$\therefore \quad 4e^{2x} - 2x - 68 = 0$

Let $f(x) = 4e^{2x} - 2x - 68$

Solve $f(x) = 0$

$f(1.4365) = -0.11296\ldots$

$f(1.4375) = 0.026696\ldots$

There is a sign change in the interval $[1.4365, 1.4375]$

$\Rightarrow \alpha \in [1.4365, 1.4375]$

$\therefore \quad \alpha \approx 1.437$ (3dp)

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5\. (i) Find, using calculus, the x coordinate of the turning point of the curve with equation

$$y = e^{3x}\cos 4x, \quad \frac{\pi}{4} \leqslant x < \frac{\pi}{2}$$

Give your answer to 4 decimal places.

(5)

(ii) Given $x = \sin^2 2y$, $0 < y < \frac{\pi}{4}$, find $\frac{dy}{dx}$ as a function of y.

Write your answer in the form

$$\frac{dy}{dx} = p\,\mathrm{cosec}(qy), \qquad 0 < y < \frac{\pi}{4}$$

where p and q are constants to be determined.

(5)

$u = e^{3x}$

5(i) $y = e^{3x}\cos 4x$ $\quad v = \cos 4x$

$\therefore \frac{dy}{dx} = (\cos 4x)(3e^{3x}) + e^{3x}(-4\sin 4x)$

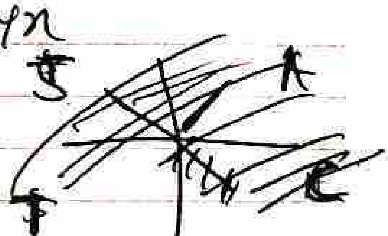
$= 3e^{3x}\cos 4x - 4e^{3x}\sin 4x$

$\frac{dy}{dx} = 0 \Rightarrow e^{3x}(3\cos 4x - 4\sin 4x) = 0$

$e^{3x} \neq 0$

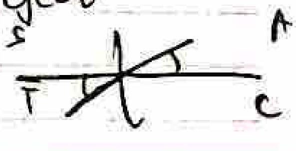
$\therefore 3\cos 4x - 4\sin 4x = 0$

$\therefore 4\sin 4x = 3\cos 4x$



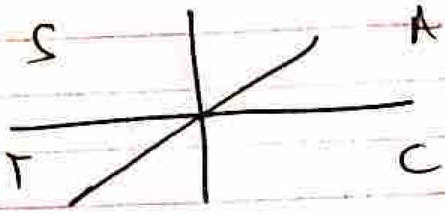
$\Rightarrow \tan 4x = \frac{3}{4}$

$\pi \leq 4x < 2\pi$ | $4x = \arctan\left(\frac{3}{4}\right) = 0.64$ x reject



Question 5 continued

~~$4x = 2\pi + 0.6435 = 6.9266\ldots$~~



$$4x = \pi + \arctan\left(\tfrac{3}{4}\right)$$

$$\Rightarrow x = \frac{\pi}{4} + \frac{1}{4}\arctan\left(\tfrac{3}{4}\right)$$

$$\therefore x = 0.9463 \ (4dp)$$

(ii) $x = \sin^2 2y$

$$\frac{dx}{dy} = 2\sin 2y \times 2\cos 2y = 4\sin 2y\cos 2y$$

$$\therefore \frac{dy}{dx} = \frac{1}{4\sin 2y\cos 2y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2(2\sin 2y\cos 2y)}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sin 4y} = \frac{1}{2}\operatorname{cosec} 4y$$

$p = \frac{1}{2} \quad q = 4$

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6. $$f(x) = \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6}, \quad x > 2, x \in \mathbb{R}$$

(a) Given that

$$\text{LHS} \quad \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6} \equiv x^2 + A + \frac{B}{x - 2} \quad \text{RHS}$$

find the values of the constants A and B.

(4)

(b) Hence or otherwise, using calculus, find an equation of the normal to the curve with equation $y = f(x)$ at the point where $x = 3$

(5)

(a)

$x > 2$:

$x = 3 \Rightarrow$ LHS $= 16$

RHS $= 9 + A + B$

$\therefore \; 9 + A + B = 16 \Rightarrow A + B = 7 \quad (1)$

$x = 4 \Rightarrow$ LHS $= 21$ RHS $= 16 + A + \frac{B}{2}$

$\therefore \; 16 + A + \frac{B}{2} = 21$

$\Rightarrow 2A + B = 10 \quad (2)$

Solve simultaneously: $A + A + B = 10$

$\therefore \; A + 7 = 10$

$\Rightarrow A = 3$

$B = 4$

$A = 3 \quad B = 4$

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Question 6 continued

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(b) $f(x) = x^2 + 3 + \frac{4}{x-2}$

$f(3) = 16$

$f'(x) = 2x - 4(x-2)^{-2}$

$f'(3) = 2$

$\therefore$ gradient of normal $= -\frac{1}{2}$

$y - y_1 = m(x - x_1)$

$\therefore \quad y - 16 = -\frac{1}{2}(x-3)$

$32 - 2y = x - 3$

$y = \frac{35 - x}{2}$

7. (a) For $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, sketch the graph of $y = g(x)$ where

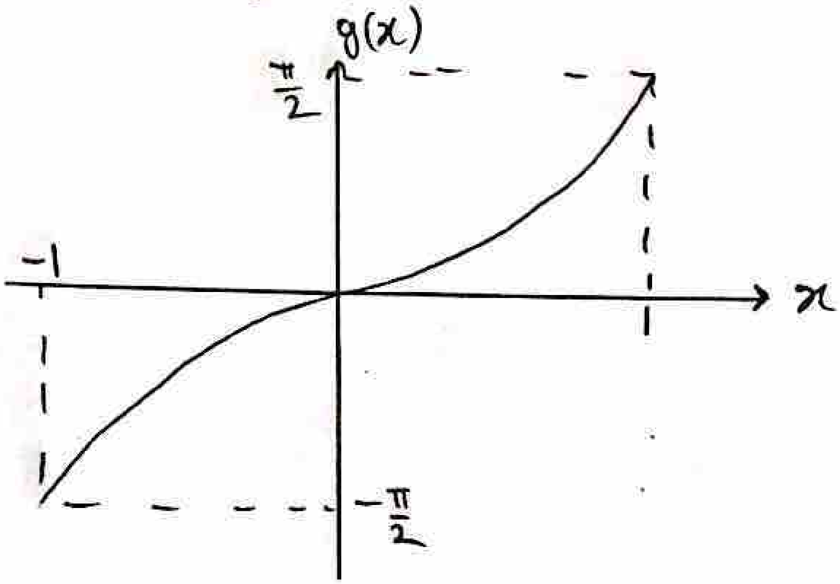
$$g(x) = \arcsin x \qquad -1 \leqslant x \leqslant 1$$

(2)

(b) Find the exact value of x for which

$$3g(x + 1) + \pi = 0$$

(3)



(b) $3g(x+1) = -\pi$

$g(x+1) = -\frac{\pi}{3}$

$g^{-1}(g(x+1)) = g^{-1}(-\frac{\pi}{3})$

$\therefore x+1 = g^{-1}(-\frac{\pi}{3})$

$\therefore x+1 = \sin(-\frac{\pi}{3})$

$x = \sin(-\frac{\pi}{3}) - 1$

$\Rightarrow x = \frac{-2-\sqrt{3}}{2}$

8. (a) Prove that

$$2\cot 2x + \tan x \equiv \cot x \qquad x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

(4)

(b) Hence, or otherwise, solve, for $-\pi \leqslant x < \pi$,

$$6\cot 2x + 3\tan x = \operatorname{cosec}^2 x - 2$$

Give your answers to 3 decimal places.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

8(a)

LHS $= 2\cot 2x + \tan x = \cancel{\frac{2\cos 2x}{\sin 2x} + \frac{\sin x}{\cos x}}$

$= \frac{2}{\tan 2x} + \tan x$

$= \frac{2}{\frac{2\tan x}{1-\tan^2 x}} + \tan x$

$= \frac{1-\tan^2 x}{\tan x} + \tan x$

$= \frac{1-\tan^2 x}{\tan x} + \frac{\tan^2 x}{\tan x}$

$= \frac{1-\tan^2 x + \tan^2 x}{\tan x} = \frac{1}{\tan x} = \cot x$

$=$ RHS

as required.

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Question 8 continued

(b) $6\cot 2x + 3\tan x = \operatorname{cosec}^2 x - 2$

$\therefore\ 3(2\cot 2x + \tan x) = \operatorname{cosec}^2 x - 2$

$\therefore\ 3\cot x = \operatorname{cosec}^2 x - 2$

$s^2 + c^2 = 1$

$\cancel{t^2 + 1 =}$

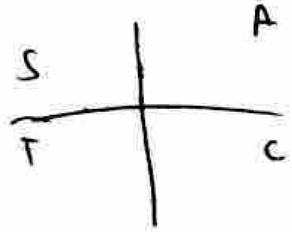
$1 + \cot^2 = \operatorname{cosec}^2$

$3\cot x = \cot^2 x - 1$

$\therefore\ \cot^2 x - 3\cot x - 1 = 0$

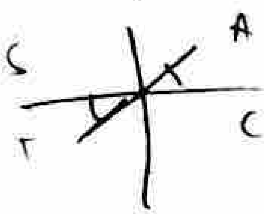
$$\cot x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ \cot x = \frac{3 \pm \sqrt{13}}{2}$$



$$\therefore\ \tan x = \frac{2}{3 \pm \sqrt{13}}$$

$$\tan x = \frac{2}{3 + \sqrt{13}}$$



$$x = \arctan\left(\frac{2}{3+\sqrt{13}}\right) = 0.294$$

$$x = -\pi + 0.294001\ldots = -2.848$$

Question 8 continued

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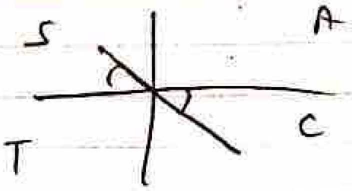
$$\tan x = \frac{2}{3-\sqrt{13}}$$



$$x = \arctan\left(\frac{2}{3-\sqrt{13}}\right) = -1.277$$

$$x = \pi - 1.27\cancel{7}6.7\ldots = 1.865$$

$$\therefore \quad x = -2.848^c \qquad x = 0.294^c$$

$$x = -1.277^c \qquad x = 1.865^c$$

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9. The amount of an antibiotic in the bloodstream, from a given dose, is modelled by the formula

$$x = De^{-0.2t}$$

where x is the amount of the antibiotic in the bloodstream in milligrams, D is the dose given in milligrams and t is the time in hours after the antibiotic has been given.

A first dose of 15 mg of the antibiotic is given.

(a) Use the model to find the amount of the antibiotic in the bloodstream 4 hours after the dose is given. Give your answer in mg to 3 decimal places. **(2)**

A second dose of 15 mg is given 5 hours after the first dose has been given. Using the same model for the second dose,

(b) show that the **total** amount of the antibiotic in the bloodstream 2 hours after the second dose is given is 13.754 mg to 3 decimal places. **(2)**

No more doses of the antibiotic are given. At time T hours after the second dose is given, the total amount of the antibiotic in the bloodstream is 7.5 mg.

(c) Show that $T = a\ln\left(b + \frac{b}{e}\right)$, where a and b are integers to be determined. **(4)**

9(a) $D=15 \quad t=4$

$x = 15e^{-0.2\times4} = 6.740$ (3dp)

(b) 7 hours since first dose:

$x = 15e^{-0.2\times7} = 3.698...$

2 hours since second dose:

$x = 15e^{-0.2\times2} = 10.054...$

$\therefore \sum x = 15e^{-0.2\times7} + 15e^{-0.2\times2}$

$= 13.7537... = 13.754$ (3dp)

as required.

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Question 9 continued

5+T hours since first dose

T hours since second dose

$\therefore \Sigma x = 15e^{-0.2(5+T)} + 15e^{-0.2T} = 7.5$

$\times e^{0.2T}$

$\Rightarrow 15e^{-0.2T-1}e^{0.2T} + 15 = 7.5e^{0.2T}$

$\therefore 15e^{-1} + 15 = 7.5e^{0.2T}$

$\div 7.5 \Rightarrow \frac{2}{e} + 2 = e^{0.2T}$

$\ln(e^{0.2T}) = \ln\left(\frac{2}{e} + 2\right)$

$\therefore 0.2T = \ln\left(\frac{2}{e} + 2\right)$

$\Rightarrow T = 5\ln\left(2 + \frac{2}{e}\right)$

$a = 5 \quad b = 2$