FP2 S15

1. Using algebra, find the set of values of x for which

$$\frac{x}{x+2} < \frac{2}{x+5}$$

(7)

$$\frac{x(x+2)^{\cancel{2}}(x+5)^2}{\cancel{x+2}} < \frac{2(x+2)^2(x+5)^{\cancel{2}}}{\cancel{x+5}}$$

$$\Rightarrow x(x+2)(x+5)^2 - 2(x+2)^2(x+5) < 0$$

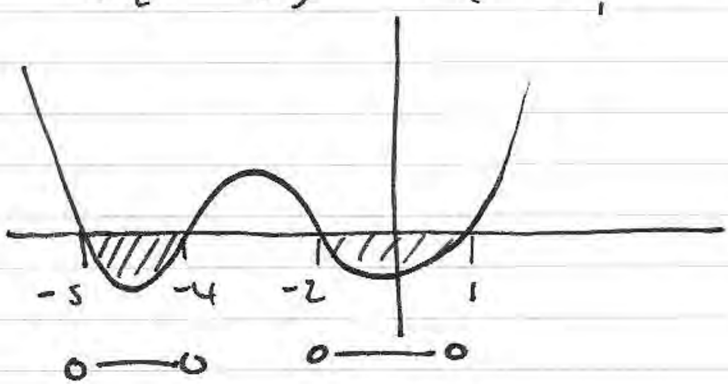
$$\Rightarrow (x+2)(x+5)[x(x+5) - 2(x+2)] < 0$$

$$\Rightarrow (x+2)(x+5)[x^2+5x-2x-4] < 0$$

$$\Rightarrow (x+2)(x+5)(x^2+3x-4) < 0$$

$$\Rightarrow (x+2)(x+5)(x+4)(x-1) < 0$$

$-2 \quad -5 \quad -4 \quad 1$



$-5 < x < -4$ or $-2 < x < 1$

2. (a) Express $\dfrac{1}{(r+6)(r+8)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+6)(r+8)} = \frac{n(an+b)}{56(n+7)(n+8)}$$

where a and b are integers to be found.

(4)

a) $\dfrac{1}{(r+6)(r+8)} = \dfrac{A}{r+6} + \dfrac{B}{r+8} \Rightarrow$ $1 = A(r+8) + B(r+6)$

$r=-6 \quad 1 = 2A \quad \therefore A = \frac{1}{2}$

$r=-8 \quad 1 = -2B \quad \therefore B = -\frac{1}{2}$

$= \dfrac{1}{2(r+6)} - \dfrac{1}{2(r+8)}$

b) $\dfrac{2}{(r+6)(r+8)} = 2\left(\dfrac{1}{2(r+6)} - \dfrac{1}{2(r+8)}\right) = \dfrac{1}{r+6} - \dfrac{1}{r+8}$

$\therefore \sum \dfrac{2}{(r+6)(r+8)} = \sum_{1}^{n} \dfrac{1}{r+6} - \dfrac{1}{r+8}$

$= \Bigg(\left(\frac{1}{7} - \cancel{\frac{1}{9}}\right) + \left(\frac{1}{8} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{11}}\right) + \ldots$

$\left(\cancel{\frac{1}{n+4}} - \cancel{\frac{1}{n+6}}\right) + \left(\cancel{\frac{1}{n+5}} - \frac{1}{n+7}\right) + \left(\cancel{\frac{1}{n+6}} - \frac{1}{n+8}\right) \Bigg)$

$$= \frac{1}{7} + \frac{1}{8} - \frac{1}{n+7} - \frac{1}{n+8}$$

$$= \frac{15}{56} - \frac{1}{n+7} - \frac{1}{n+8}$$

$$= \frac{15(n+7)(n+8) - 56(n+8) - 56(n+7)}{56(n+7)(n+8)}$$

$$= \frac{15n^2 + 225n + 84\cancel{0} - 56n - 4\cancel{4}8 - 56n - 39\cancel{2}}{56(n+7)(n+8)}$$

$$= \frac{15n^2 + 113n}{56(n+7)(n+8)} = \frac{n(15n+113)}{56(n+7)(n+8)}$$

3. (a) Show that the substitution $z = y^{-2}$ transforms the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \quad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} - 4xz = -2xe^{-x^2} \quad \text{(II)}$$

(4)

(b) Solve differential equation (II) to find z as a function of x.

(5)

(c) Hence find the general solution of differential equation (I), giving your answer in the form $y^2 = f(x)$.

(1)

$$z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \quad \therefore \frac{dy}{dx} = -\frac{1}{2}y^3\frac{dz}{dx}$$

$$\therefore \frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \qquad z = \frac{1}{y^2} \Rightarrow y^2 = \frac{1}{z} \Rightarrow y = \frac{1}{z^{\frac{1}{2}}}$$

$$= -\frac{1}{2}y^3\frac{dz}{dx} + 2xz^{-\frac{1}{2}} = xe^{-x^2}z^{-\frac{3}{2}}$$

$$\frac{-\frac{1}{2}z^{-\cancel{\frac{3}{2}}}}{\cancel{z^{-3/2}}}\frac{dz}{dx} + \frac{2xz^{-\frac{1}{2}}}{z^{-3/2}} = \frac{xe^{-x^2}\cancel{z^{-\frac{3}{2}}}}{\cancel{z^{-\frac{3}{2}}}}$$

$$-\frac{1}{2}\frac{dz}{dx} + 2xz = xe^{-x^2} \qquad (\times -2)$$

$$\therefore \frac{dz}{dx} - 4xz = -2xe^{-x^2}$$

☆

b) $IF = e^{-\int 4x\,dx} = e^{-2x^2}$

$$e^{-2x^2}\frac{dz}{dx} - 4xe^{-2x^2}z = -2xe^{-x^2} \times e^{-2x^2}$$

$$\frac{d}{dx}\left[e^{-2x^2}z\right] = -2xe^{-3x^2}$$

$$e^{-2x^2}z = \int -2xe^{-3x^2}\,dx$$

$$= \int e^{-3x^2}(-2x\,dx)$$

$$= \int u \times \frac{1}{3u}\,du$$

$u = e^{-3x^2}$

$\frac{du}{dx} = -6xe^{-3x^2}$

$du = 3e^{-3x^2}(-2x\,dx)$

$\frac{1}{3u}\,du = -2x\,dx$

$$\therefore e^{-2x^2}z = \frac{1}{3}u + c$$

$$\therefore e^{-2x^2}z = \frac{1}{3}e^{-3x^2} + c$$

$$\therefore z = \frac{1}{3}e^{-3x^2} \times e^{2x^2} + ce^{2x^2}$$

$$\therefore z = \frac{1}{3}e^{-x^2} + ce^{2x^2}$$

c) $\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + ce^{2x^2}$ $\quad\therefore y^2 = \dfrac{1}{\frac{1}{3}e^{-x^2} + ce^{2x^2}}$

4. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z-1}{z+1}, \quad z \neq -1$$

The line in the z-plane with equation $y = 2x$ is mapped by T onto the curve C in the w-plane.

(a) Show that C is a circle and find its centre and radius.

(7)

The region $y < 2x$ in the z-plane is mapped by T onto the region R in the w-plane.

(b) Sketch circle C on an Argand diagram and shade and label region R.

(2)

$w(z+1) = z-1 \Rightarrow wz + w = z - 1 \Rightarrow wz - z = -1 - w$

$\Rightarrow z - wz = w + 1$

$\Rightarrow z(1-w) = w+1 \quad \therefore z = \dfrac{w+1}{1-w} \qquad w = u + iv$

$$\therefore z = \frac{(u+1)+iv}{(1-u)-iv} \times \frac{[(1-u)+iv]}{[(1-u)+iv]} = \frac{(u+1)(1-u) - v^2 + i(v(1-u) + v(u+1))}{(1-u)^2 + v^2}$$

$$\therefore z = \frac{(1-u^2-v^2) + i(v - uv + uv + v)}{(1-u)^2+v^2} \equiv x + iy$$

$$\therefore x = \frac{1-u^2-v^2}{(1-u)^2+v^2} \qquad y = \frac{2v}{(1-u)^2+v^2}$$

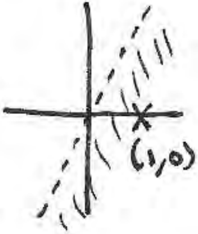
$$y = 2x \quad \therefore \frac{\cancel{2}v}{(1-u)^2+v^2} = \frac{\cancel{2}(1-u^2-v^2)}{(1-u^2)+v^2}$$

$$\therefore v = 1 - u^2 - v^2 \Rightarrow v^2 + v + u^2 = 1$$

$$\Rightarrow \left(v+\tfrac{1}{2}\right)^2 + u^2 = 1 + \tfrac{1}{4} = \tfrac{5}{4} = \left(\tfrac{\sqrt{5}}{2}\right)^2 \quad \therefore u^2 + \left(v + \tfrac{1}{2}\right)^2 = \left(\tfrac{\sqrt{5}}{2}\right)^2$$

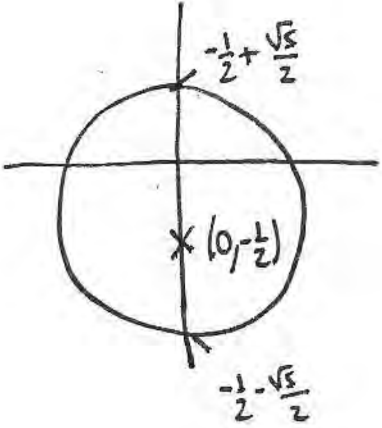
$C\left(0, -\tfrac{1}{2}\right) \quad r = \dfrac{\sqrt{5}}{2}$

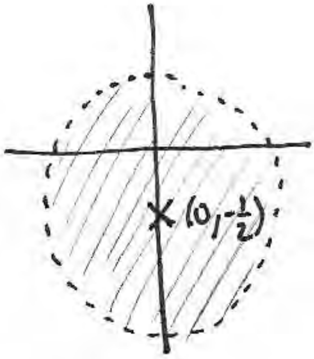
b) $y < 2x$



$z = 1 + 0i$

$$w = \frac{1 - 1 + 0i}{1 + 1 + 0i} = \frac{0}{2} + 0i \Rightarrow (0,0)$$





5. Given that $y = \cot x$,

(a) show that

$$\frac{d^2y}{dx^2} = 2\cot x + 2\cot^3 x$$

(3)

(b) Hence show that

$$\frac{d^3y}{dx^3} = p\cot^4 x + q\cot^2 x + r$$

where p, q and r are integers to be found.

(3)

(c) Find the Taylor series expansion of $\cot x$ in ascending powers of $\left(x - \frac{\pi}{3}\right)$ up to and including the term in $\left(x - \frac{\pi}{3}\right)^3$.

(3)

a) $y = \cot x \Rightarrow \frac{dy}{dx} = -\operatorname{cosec}^2 x = -(\operatorname{cosec} x)^2$

$\Rightarrow \frac{d^2y}{dx^2} = -2(\operatorname{cosec} x)^1 \times -\operatorname{cosec} x \cot x$

$= +2\operatorname{cosec}^2 x \cot x$

$\frac{\sin^2}{\sin^2} + \frac{\cos^2}{\sin^2} = \frac{1}{\sin^2} \Rightarrow 1 + \cot^2 = \operatorname{cosec}^2 x$

$\therefore \frac{d^2y}{dx^2} = +2\cot x(1+\cot^2)$

$= 2\cot x + 2\cot^3 x$

b) $\frac{d^3y}{dx^3} = -2\operatorname{cosec}^2 x + 6(\cot x)^2 \times -\operatorname{cosec}^2 x$

$= -2\operatorname{cosec}^2 x\,(1 + 3\cot^2 x)$

$= -2(1+\cot^2 x)(1+3\cot^2 x)$

$= -6\cot^4 x - 8\cot^2 x - 2$

$$\left(x - \frac{\pi}{3}\right)$$

$$\cot\frac{\pi}{3} = \frac{1}{\tan\frac{\pi}{3}} = \frac{1}{\sqrt{3}} \qquad \cot^2\left(\frac{\pi}{3}\right) = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$\operatorname{cosec}\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\operatorname{cosec}^2\left(\frac{\pi}{3}\right) = \frac{4}{3}$$

$$\therefore \quad f\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$f'\left(\frac{\pi}{3}\right) = -\frac{4}{3}$$

$$f''\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}} + 2\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{2^{\times 3}}{\sqrt{3}_{\times 3}} + \frac{2}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} = \frac{8}{9}\sqrt{3}$$

$$f'''\left(\frac{\pi}{3}\right) = -6\left(\frac{1}{\sqrt{3}}\right)^4 - 8\left(\frac{1}{\sqrt{3}}\right)^2 - 2 = -\frac{6}{9} - \frac{8^{\times 3}}{3_{\times 3}} - \frac{2^{\times 9}}{1_{\times 9}} = \frac{-48}{9} = -\frac{16}{3}$$

$$f(x) = \frac{\sqrt{3}}{3} - \frac{4}{3}\left(x - \frac{\pi}{3}\right) + \frac{4}{9}\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 - \frac{8}{9}\left(x - \frac{\pi}{3}\right)^3$$

6. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2\sin x \quad \text{(I)}$$

(8)

Given that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$

(b) find the particular solution of differential equation (I).

(5)

CF $y = Ae^{mx}$ $\quad y'' - 2y' - 3y = 0$

$y' = Ame^{mx}$ $\quad Am^2e^{mx} - 2Ame^{mx} - 3Ae^{mx} = 0$

$y'' = Am^2e^{mx}$ $\quad Ae^{mx}(m^2 - 2m - 3) = 0$

$\neq 0 \quad (m-3)(m+1) = 0$

$m = 3 \quad m = -1$

$\therefore y_{CF} = Ae^{3x} + Be^{-x}$

PI $y = P\sin x + Q\cos x$ $\quad y'' - 2y' - 3y \equiv 2\sin x$

$y' = P\cos x - Q\sin x$ $\quad -P\sin x - Q\cos x$

$y'' = -P\sin x - Q\cos x$ $\quad +2Q\sin x - 2P\cos x$

$-3P\sin x - 3Q\cos x$

$(-4P + 2Q)\sin x + (-4Q - 2P)\cos x \equiv 2\sin x$

$-4P + 2Q = 2 \Rightarrow -4P + 2Q = 2 \quad \therefore Q = \frac{1}{5}$

$2P + 4Q = 0 \quad 4P + 8Q = 0 \; +$

$10Q = 2 \quad \therefore P = -\frac{2}{5}$

$2P = -\frac{4}{5}$

$y_{PI} = -\frac{2}{5}\sin x + \frac{1}{5}\cos x$

$\therefore y_{GS} = Ae^{3x} + Be^{-x} - \frac{2}{5}\sin x + \frac{1}{5}\cos x$

$y = 0$ when $x = 0$ $\qquad 0 = A + B + \frac{1}{5} \Rightarrow 5A + 5B = -1$

$y' = 1$ when $x = 0$

$$y' = 3Ae^{3x} - Be^{-x} - \frac{2}{5}\cos x - \frac{1}{5}\sin x$$

$$1 = 3A - B - \frac{2}{5} \qquad 3A - B = \frac{7}{5}$$

$$\begin{aligned} 15A - 5B &= 7 \\ 5A + 5B &= -1 \quad + \\ \hline 20A &= 6 \quad A = \frac{3}{10} \end{aligned}$$

$$\frac{15}{10} = -1 - 5B$$

$$\frac{5}{2} = -5B \qquad B = -\frac{1}{2}$$

$$\therefore \quad y = \frac{3}{10}e^{3x} - \frac{1}{2}e^{-x} - \frac{2}{5}\sin x + \frac{1}{5}\cos x$$

7.

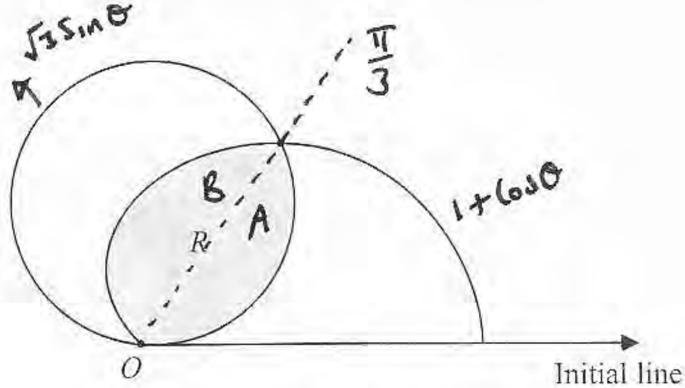


Figure 1

Figure 1 shows the two curves given by the polar equations

$$r = \sqrt{3}\sin\theta, \quad 0 \leqslant \theta \leqslant \pi$$

$$r = 1 + \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$$

(a) Verify that the curves intersect at the point P with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{3}\right)$. **(2)**

The region R, bounded by the two curves, is shown shaded in Figure 1.

(b) Use calculus to find the exact area of R, giving your answer in the form $a\left(\pi - \sqrt{3}\right)$, where a is a constant to be found. **(6)**

a) $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ $\quad r = \frac{3}{2}$ $\quad \frac{3}{2} = \sqrt{3}\sin\theta \Rightarrow \sin\theta = \frac{3}{2\sqrt{3}}$

$\Rightarrow \sin\theta = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \quad \therefore \theta = \frac{\pi}{3}$ ✓

Area $= \frac{1}{2}\int r^2 \, d\theta$

$\frac{3}{2} = 1 + \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ ✓

b) $R = A + B$ $\quad A = \int_0^{\frac{\pi}{3}} \frac{1}{2}(\sqrt{3}\sin\theta)^2 \, d\theta = \frac{3}{2}\int_0^{\frac{\pi}{3}} \sin^2\theta \, d\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$ $\quad = \frac{3}{4}\int_0^{\frac{\pi}{3}} 1 - \cos 2\theta \, d\theta$

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$= \frac{3}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}$

$= \frac{3}{4}\left[\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) - (0 - 0)\right] = \frac{3\pi}{12} - \frac{3\sqrt{3}}{16} = \frac{\pi}{4} - \frac{3\sqrt{3}}{16}$

$$B = \int_{\frac{\pi}{3}}^{\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} \cos^2\theta + 2\cos\theta + 1\,d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\tfrac{1}{2}\cos 2\theta + \frac{1}{2}$

$$\Rightarrow \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} \tfrac{1}{2}\cos 2\theta + 2\cos\theta + \frac{3}{2}\,d\theta$$

$$\Rightarrow \frac{1}{4}\int_{\frac{\pi}{3}}^{\pi} \cos 2\theta + 4\cos\theta + 3\,d\theta = \frac{1}{4}\left[\tfrac{1}{2}\sin 2\theta + 4\sin\theta + 3\theta\right]_{\frac{\pi}{3}}^{\pi}$$

$$= \frac{1}{8}\left[\sin 2\theta + 8\sin\theta + 6\theta\right]_{\frac{\pi}{3}}^{\pi}$$

$$= \frac{1}{8}\left[(0+0+6\pi) - \left(\frac{\sqrt{3}}{2} + 4\sqrt{3} + 2\pi\right)\right]$$

$$= \frac{1}{8}\left[6\pi - 2\pi - \frac{9\sqrt{3}}{2}\right) = \frac{\pi}{2} - \frac{9}{16}\sqrt{3}$$

$$\therefore R = \frac{\pi}{4} + \frac{\pi}{2} - \left(\frac{3\sqrt{3}}{16} + \frac{9\sqrt{3}}{16}\right)$$

$$= \frac{3}{4}\pi - \frac{3}{4}\sqrt{3} = \frac{3}{4}(\pi - \sqrt{3})$$

8. (a) Show that

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = z^6-\frac{1}{z^6}-k\left(z^2-\frac{1}{z^2}\right)$$

where k is a constant to be found.

(3)

Given that $z = \cos\theta + \mathrm{i}\sin\theta$, where θ is real,

(b) show that

(i) $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

(ii) $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta$

(3)

(c) Hence show that

$$\cos^3\theta\,\sin^3\theta = \frac{1}{32}\,(3\sin 2\theta - \sin 6\theta)$$

(4)

(d) Find the exact value of

$$\int_0^{\frac{\pi}{8}} \cos^3\theta\,\sin^3\theta\,\mathrm{d}\theta$$

(4)

$(z+z^{-1})^3 = z^3 + 3z^2z^{-1} + 3zz^{-2} + z^{-3}$

$= z^3 + 3z + 3z^{-1} + z^{-3}$

$(z-z^{-1})^3 = z^3 - 3z + 3z^{-1} - z^{-3}$

	z^3	$+3z$	$+3z^{-1}$	$+z^{-3}$
z^3	z^6	$\cancel{3z^4}$	$3z^2$	$\cancel{1}$
$-3z$	$\cancel{-3z^4}$	$-9z^2$	$\cancel{-9}$	$-3z^{-2}$
$3z^{-1}$	$3z^2$	$\cancel{9}$	$9z^{-2}$	$\cancel{3z^{-4}}$
$-z^{-3}$	$\cancel{-1}$	$-3z^{-2}$	$\cancel{-3z^{-4}}$	$-z^{-6}$

$= z^6 - 3z^2 + 3z^{-2} - z^{-6}$

$= z^6 - \dfrac{1}{z^6} - 3\left(z^2 - \dfrac{1}{z^2}\right)$

b) $z = \cos\theta + i\sin\theta$

$$z^n = \cos n\theta + i\sin n\theta \quad (\text{DEM})$$
$$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$
$$z^{-n} = \cos(n\theta) - i\sin(n\theta)$$
$$\underline{z^n = \cos(n\theta) + i\sin(n\theta)} \; + \quad b) \uparrow -$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad z^n - \frac{1}{z^n} = +2i\sin(n\theta)$$

c) $z + \frac{1}{z} = 2\cos\theta \qquad z - \frac{1}{z} = -2i\sin\theta$

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = (2\cos\theta)^3(+2i\sin\theta)^3$$
$$= 2^3((\cos\theta)^3(+2i)^3(\sin\theta)^3$$
$$= -64i\,((\cos\theta)^3(\sin\theta)^3$$

$$\therefore -64i\,((\cos\theta)^3(\sin\theta)^3 = z^6 - \frac{1}{z^6} - 3\left(z^2 - \frac{1}{z^2}\right)$$

$$-64i\,(\cos\theta)^3(\sin\theta)^3 = (2i\sin 6\theta) - 3(2i\sin 2\theta)$$

$\div -2i \qquad 32(\cos\theta)^3(\sin\theta)^3 = -\sin 6\theta + 3\sin 2\theta$

$$\therefore \cos^3\theta\sin^3\theta = \frac{1}{32}(3\sin 2\theta - \sin 6\theta)$$

d) $\int_0^{\frac{\pi}{8}} \cos^3\theta\sin^3\theta\, d\theta = \frac{1}{32}\int_0^{\frac{\pi}{8}} 3\sin 2\theta - \sin 6\theta\, d\theta$

$$= \frac{1}{32}\left[-\frac{3}{2}\cos 2\theta + \frac{1}{6}\cos 6\theta\right]_0^{\frac{\pi}{8}} = \frac{1}{192}\left[\cos 6\theta - 9\cos 2\theta\right]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{192}\left[\left(-\frac{\sqrt{2}}{2} - \frac{9\sqrt{2}}{2}\right) - (1-9)\right] = \frac{1}{192}(8 - 5\sqrt{2})$$