S2 S14 UK

1. Patients arrive at a hospital accident and emergency department at random at a rate of 6 per hour.

(a) Find the probability that, during any 90 minute period, the number of patients arriving at the hospital accident and emergency department is

(i) exactly 7

(ii) at least 10

(5)

A patient arrives at 11.30 a.m.

(b) Find the probability that the next patient arrives before 11.45 a.m.

(3)

a) x = # patients arriving per 90min

$x \sim Po(9)$

i) $P(x=7) = \dfrac{e^{-9} \times 9^7}{7!} = 0.1171$

2

ii) $P(x \geq 10) = 1 - P(x \leq 9) = 1 - 0.5874 = 0.4126$

$P(x > 9)$

2

b) y = # patients arriving per 15min

$y \sim Po(1.5)$

$P(y \geq 1) = 1 - P(y=0) = 1 - e^{-1.5} = 0.7769$

2

2. The length of time, in minutes, that a customer queues in a Post Office is a random variable, T, with probability density function

$$f(t) = \begin{cases} c(81 - t^2) & 0 \leqslant t \leqslant 9 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

(a) Show that the value of c is $\frac{1}{486}$ **(4)**

(b) Show that the cumulative distribution function $F(t)$ is given by

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{6} - \frac{t^3}{1458} & 0 \leqslant t \leqslant 9 \\ 1 & t > 9 \end{cases}$$

(2)

(c) Find the probability that a customer will queue for longer than 3 minutes. **(2)**

A customer has been queueing for 3 minutes.

(d) Find the probability that this customer will be queueing for at least 7 minutes. **(3)**

Three customers are selected at random.

(e) Find the probability that exactly 2 of them had to queue for longer than 3 minutes. **(3)**

a) $c\int_0^9 81 - t^2\,dt = 1 \quad c\left[81t - \frac{1}{3}t^3\right]_0^9 = 1$

$\Rightarrow c\left[486 - 0\right] = 1 \Rightarrow 486c = 1 \quad \therefore c = \frac{1}{486}$ #

b) $F(t) = \int f(t)\,dt = \frac{1}{486}\int_0^t 81 - x^2\,dx = \frac{1}{486}\left[81x - \frac{1}{3}x^3\right]_0^t$

$= \frac{1}{486}(81t - \frac{1}{3}t^3) = \frac{t}{6} - \frac{t^3}{1458}$

$P(t < 0) = 0$

$P(t > 9) = 1$

$\therefore F(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{6} - \frac{t^3}{1458} & 0 \leq t \leq 9 \\ 1 & t > 9 \end{cases}$

c) $P(t>3) = 1-F(3) = \frac{14}{27}$

d) $P(t>7 \mid t>3) = \frac{P(t>7)}{P(t>3)} = \frac{1-F(7)}{1-F(3)} = \frac{1-\frac{679}{729}}{\frac{14}{27}}$

$= 0.1323$

e) x = # customers waiting longer than 3 min

$x \sim B(3, \frac{14}{27})$ $\quad P(X=2) = {}^3C_2\left(\frac{14}{27}\right)^2\left(\frac{13}{27}\right) = 0.3884$

3. A company claims that it receives emails at a mean rate of 2 every 5 minutes.

(a) Give two reasons why a Poisson distribution could be a suitable model for the number of emails received.
(2)

(b) Using a 5% level of significance, find the critical region for a two-tailed test of the hypothesis that the mean number of emails received in a 10 minute period is 4. The probability of rejection in each tail should be as close as possible to 0.025
(2)

(c) Find the actual level of significance of this test.
(2)

To test this claim, the number of emails received in a random 10 minute period was recorded.

During this period 8 emails were received.

(d) Comment on the company's claim in the light of this value. Justify your answer.
(2)

During a randomly selected 15 minutes of play in the Wimbledon Men's Tennis Tournament final, 2 emails were received by the company.

(e) Test, at the 10% level of significance, whether or not the mean rate of emails received by the company during the Wimbledon Men's Tennis Tournament final is lower than the mean rate received at other times. State your hypotheses clearly.
(5)

a) emails are received randomly, independently singularly and a constant rate

b) $x = \#$ emails per 10min $\quad x \sim Po(4)$

$H_0: \lambda = 4 \quad P(x \leq L) \simeq 0.025 \quad P(x \geq u) \simeq 0.025$

$H_1: \lambda \neq 4 \quad P(x \leq 0) = 0.0183 * \quad P(x > u-1) = 1 - P(x \leq u-1) \simeq 0.025$

$P(x \leq 1) = 0.0916 \quad P(x \leq u-1) \simeq 0.975$

$L = 0 \quad P(x \leq 7) = 0.9489$

$P(x \leq 8) = 0.9786 *$

$CR \; \{x = 0\} \cup \{x \geq 9\} \quad u - 1 = 8 \;\therefore u = 9$

c) $ASL = \begin{array}{r} 0.0183 \\ +\;0.0214 \\ \hline 0.0397 \end{array} \quad \therefore 3.97\%$

2

d) 8 emails doesn't fall in the critical region
∴ result is not statistically significant
∴ not enough evidence to reject null hypothesis
∴ evidence to support claim of 2 per 5min.

e) y = # emails per 15min $\quad y \sim Po(6)$

$H_0: \lambda = 6 \quad P(x \leq 2) = 0.0620 \quad < 10\%$
$H_1: \lambda < 6$

∴ result is statistically significant at the 10% CI level
∴ enough evidence to reject null hypothesis and accept alternate hypothesis
∴ evidence suggests mean rate of emails is lower during the tennis final.

4. A cadet fires shots at a target at distances ranging from 25 m to 90 m. The probability of hitting the target with a single shot is p. When firing from a distance d m, $p = \frac{3}{200}(90 - d)$.

Each shot is fired independently.

The cadet fires 10 shots from a distance of 40 m.

(a) (i) Find the probability that exactly 6 shots hit the target.

(ii) Find the probability that at least 8 shots hit the target.

(5)

The cadet fires 20 shots from a distance of x m.

(b) Find, to the nearest integer, the value of x if the cadet has an 80% chance of hitting the target at least once.

(4)

The cadet fires 100 shots from 25 m.

(c) Using a suitable approximation, estimate the probability that at least 95 of these shots hit the target.

(5)

a) i) $d = 40 \Rightarrow p = 0.75$ $\quad X \sim B(10, 0.75)$

$X = \#$ hits

$P(X=6) = {}^{10}C_6\, 0.75^6\, 0.25^4$

$= 0.1460$

ii) $P(X \geq 8) = P(X=8) + P(X=9) + P(X=10)$

$= {}^{10}C_8\, 0.75^8\, 0.25^2 + {}^{10}C_9\, 0.75^9\, 0.25 + 0.75^{10}$

$= 0.5256$

alt $\quad y = \#$misses $\quad y \sim B(10, 0.25)$

$P(X \geq 8) = P(y \leq 2) = 0.5256$

b) $J = \#$ hits per 20 shots from x m $\quad J \sim B(20, p)$

$P(J \geq 1) = 0.80 \Rightarrow 1 - P(J=0) = 0.8$

$\Rightarrow P(J=0) = 0.2 \quad (1-p)^{20} = 0.2 \Rightarrow 1 - p = \sqrt[20]{0.2}$

$\therefore p = 1 - \sqrt[20]{20} = 0.0773$

$\Rightarrow 0.07731916\ldots = \frac{3}{200}(90-d) \quad \therefore 90-d = 5.1546$

$\therefore d = 84.845$

$\underline{d \simeq 85m}$

2

c) V = # hits from 100 shots at 25m
$V \sim B(100, 0.975)$

W = # misses from 100 shots at 25m
$W \sim B(100, 0.025)$

$P(V \geq 95) = P(W \leq 5)$

$np = 2.5$

$\approx W \sim Po(2.5)$

$P(W \leq 5) = \underline{0.9580}$

2

$np < 10$ so Poisson should be suitable.

$\therefore P(V \geq 95) = \underline{0.9580}$

2

<u>alt</u>

using normal approx. I don't think this is suitable since $p \not\approx 0.5$ and 0.975 is very high!

$P(V \geq 95) \overset{cc}{\rightarrow} P(V > 94.5)$

$P(V > 94)$

$\mu = np = 97.5$

$\sigma^2 = np(1-p) = 2.4375$

$\approx V \sim N(97.5, 2.4375)$

$\approx P\left(Z > \frac{94.5 - 97.5}{\sqrt{2.4375}}\right) \approx P(Z > -1.92)$

-1.92

$= \Phi(1.92) = 0.9726$

5. (a) State the conditions under which the normal distribution may be used as an approximation to the binomial distribution.
(2)

A company sells seeds and claims that 55% of its pea seeds germinate.

(b) Write down a reason why the company should not justify their claim by testing all the pea seeds they produce.
(1)

To test the company's claim, a random sample of 220 pea seeds was planted.

(c) State the hypotheses for a two-tailed test of the company's claim.
(1)

Given that 135 of the 220 pea seeds germinated,

(d) use a normal approximation to test, at the 5% level of significance, whether or not the company's claim is justified.
(7)

a) p is close to 0.5 $\quad X \sim B(n, p) \approx X \sim N(\mu, \sigma^2)$
n is large $\quad \mu = np$
$\sigma^2 = np(1-p)$

b) the test could damage the seeds, the company needs to sell some seeds.

c) $H_0 : p = 0.55 \quad x = \#$ seeds that germinate
$H_1 : p \neq 0.55 \quad x \sim B(220, 0.55)$

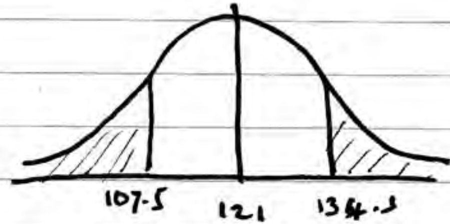
$\mu = np = 121$
$\sigma^2 = np(1-p) = 54.45$

d) $P(x \geq 135)$ cc $P(x \geq 134.5)$
$P(x > 134)$

$\approx P\left(Z > \frac{134.5 - 121}{\sqrt{54.45}}\right) \approx P(Z > 1.83) = 1 - \Phi(1.83)$
$= 0.0336$

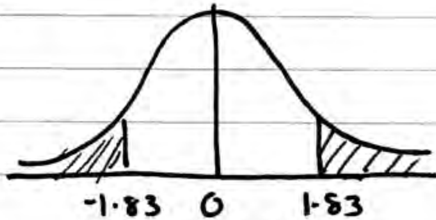
5% SL on a 2-tail test, each tail must be < 2.5% for result to be significant
$\therefore 0.0336 > 0.025$ so result is not significant
$\therefore$ not enough evidence to reject null hypothesis
$\therefore$ enough evidence to support claim that $p = 0.55$

alt



reject null if
$P(x > 134.5) + P(x < 107.5)$
$< 5\%$

-121
$\div \sqrt{54.45}$



$P(Z > 1.83) + P(Z < -1.83)$

$= 2(1 - \Phi(1.83))$

$= 0.0672$

$> 5\%$ $\therefore$ not significant.

*alt 1 tail test - I doubt this will be accepted.

$P(Z > 1.83) = 1 - \Phi(1.83) = 0.0336 < 5\%$

$\therefore$ result is significant

$\therefore$ reject null hypothesis

$\therefore$ evidence suggests $p > 0.55$.

6. The continuous random variable X has probability density function f(x) given by

$$f(x) = \begin{cases} \frac{2x}{9} & 0 \leqslant x \leqslant 1 \\ \frac{2}{9} & 1 < x < 4 \\ \frac{2}{3} - \frac{x}{9} & 4 \leqslant x \leqslant 6 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find E(X).
(4)

(b) Find the cumulative distribution function F(x) for all values of x.
(6)

(c) Find the median of X.
(3)

(d) Describe the skewness. Give a reason for your answer.
(2)

$E(x) = \int x f(x)\,dx$

$E(x) = \int_0^1 \frac{2}{9}x^2\,dx + \int_1^4 \frac{2}{9}x\,dx + \int_4^6 \frac{2}{3}x - \frac{x^2}{9}\,dx$

$= \underline{2.78}$ $\quad = \left[\frac{2}{27}x^3\right]_0^1 + \left[\frac{1}{9}x^2\right]_1^4 + \left[\frac{1}{3}x^2 - \frac{x^3}{27}\right]_4^6$

$E(X) = \underline{2.78}$

b) $0 \leq x \leq 1$ $\quad F(x) = \int_0^x \frac{2}{9}t\,dt = \left[\frac{1}{9}t^2\right]_0^x = \frac{1}{9}x^2$

$1 < x < 4$ $\quad F(x) = F(1) + \int_1^x \frac{2}{9}\,dt = \frac{1}{9} + \left[\frac{2}{9}t\right]_1^x = \frac{2}{9}x - \frac{1}{9}$

$4 < x \leq 6$ $\quad F(x) = F(4) + \int_4^x \frac{2}{3} - \frac{t}{9}\,dt = \frac{7}{9} + \left[\frac{2}{3}t - \frac{t^2}{18}\right]_4^x$

$= \frac{2x}{3} - \frac{x^2}{18} - 1$

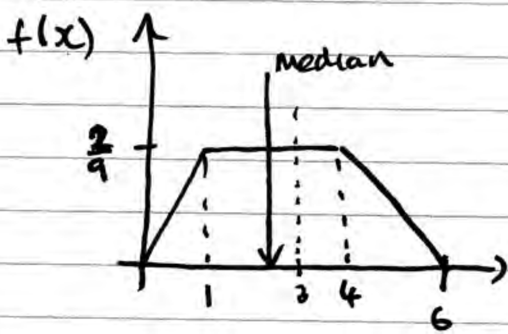
$$\therefore F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{9}x^2 & 0 \leq x \leq 1 \\ \frac{2}{9}x - \frac{1}{9} & 1 < x < 4 \\ \frac{2}{3}x - \frac{x^2}{18} - 1 & 4 \leq x \leq 6 \\ 1 & x > 6 \end{cases}$$

c) $F(\text{median}) = \frac{1}{2}$ $\quad F(1) = \frac{1}{9}$ $\quad F(4) = \frac{7}{9}$ $\quad \therefore 1 \leq \text{median} \leq 4$

$\frac{2}{9}x - \frac{1}{9} = \frac{1}{2} \Rightarrow \frac{2}{9}x = \frac{11}{18} \Rightarrow x = \text{median} = \underline{2.75}$

d) Mean > Median $\therefore$ V. slight positive skew
(2.78) (2.75) almost symmetrical

aii



Median is slightly to the left of middle
$\therefore$ Slight positive skew from shape of graph