1. The line l passes through the point $P(2, 1, 3)$ and is perpendicular to the plane Π whose vector equation is

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 3$$

Find

(a) a vector equation of the line l, **(2)**

(b) the position vector of the point where l meets Π. **(4)**

(c) Hence find the perpendicular distance of P from Π. **(2)**

a) $(\underline{i} - 2\underline{j} - \underline{k})$ IS NORMAL TO Π, HENCE A VECTOR EQUATION OF l IS

$$\underline{r} = (2\underline{i} + \underline{j} + 3\underline{k}) + \lambda(\underline{i} - 2\underline{j} - \underline{k})$$

b) SINCE THE POINT LIES ON l, $\underline{r} = \begin{pmatrix} 2+\lambda \\ 1-2\lambda \\ 3-\lambda \end{pmatrix}$

SINCE THE POINT ALSO LIES ON Π, THEN

$$\begin{pmatrix} 2+\lambda \\ 1-2\lambda \\ 3-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 3$$

$\Rightarrow (2+\lambda) - 2(1-2\lambda) - (3-\lambda) = 3$

$\Rightarrow -3 + 6\lambda = 3 \Rightarrow 6\lambda = 6 \Rightarrow \lambda = 1$

HENCE POSITION VECTOR OF INTERSECTION IS $\begin{pmatrix} 2+1 \\ 1-2 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

c) DISTANCE BETWEEN $(2, 1, 3)$ AND $(3, -1, 2)$ IS

$\sqrt{(2-3)^2 + (1-(-1))^2 + (3-2)^2} = \sqrt{1+4+1}$

$= \sqrt{6}$

Leave blank

2.
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}$$

(a) Show that matrix **M** is not orthogonal.
(2)

(b) Using algebra, show that 1 is an eigenvalue of **M** and find the other two eigenvalues of **M**.
(5)

(c) Find an eigenvector of **M** which corresponds to the eigenvalue 1
(2)

The transformation $M : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix **M**.

(d) Find a cartesian equation of the image, under this transformation, of the line

$$x = \frac{y}{2} = \frac{z}{-1}$$

(4)

a) CONSIDER SECOND AND THIRD COLUMNS (FOR EXAMPLE), TAKING THE SCALAR PRODUCT GIVES

$$\begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 0 + 4 + 0 = 4$$

SINCE THIS IS NOT ZERO, $\underline{M}$ IS NOT ORTHOGONAL.

b) ~~[illegible]~~ WE MUST SOLVE $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 4-\lambda & 1 \\ 0 & 5 & -\lambda \end{vmatrix} = 0.$

EXPAND DOWN FIRST COLUMN,

$$(1-\lambda)\big((4-\lambda)(-\lambda) - 5\big) + 0 + 0 = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 4\lambda - 5) = 0 \Rightarrow (1-\lambda)(\lambda-5)(\lambda+1) = 0$$

HENCE EIGENVALUES ARE $\lambda = 1$, $\lambda = 5$ AND $\lambda = -1$ //

Question 2 continued

c) WE MUST FIND A VECTOR $\underline{v}$ SUCH THAT $\underline{M}\,\underline{v} = \underline{v}$.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \left.\begin{aligned} x \quad + 2z &= x \\ 4y + z &= y \\ 5y \quad &= z \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} 2z &= 0 \\ z &= -3y \\ z &= 5y \end{aligned}\right\} \Rightarrow y = z = 0$$

HENCE A SUITABLE EIGENVECTOR WOULD BE $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

d) SETTING $\mu = x = \frac{y}{2} = \frac{z}{-1}$ GIVES

$$\left.\begin{aligned} x &= \mu \\ y &= 2\mu \\ z &= -\mu \end{aligned}\right\}$$

SO $\underline{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} \mu \\ 2\mu \\ -\mu \end{pmatrix} = \begin{pmatrix} \mu - 2\mu \\ 8\mu - \mu \\ 10\mu \end{pmatrix} = \begin{pmatrix} -\mu \\ 7\mu \\ 10\mu \end{pmatrix}$

GIVING $\mu = \frac{x}{-1} = \frac{y}{7} = \frac{z}{10}$

HENCE AN EQUATION OF THE IMAGE IS

$$\frac{x}{-1} = \frac{y}{7} = \frac{z}{10}$$

3. Using calculus, find the exact value of

(a) $\int_1^2 \frac{1}{\sqrt{(x^2-2x+3)}}\,dx$ **(4)**

(b) $\int_0^1 e^{2x}\sinh x\,dx$ **(4)**

a) $\int_1^2 \frac{1}{\sqrt{x^2-2x+3}}\,dx = \int_1^2 \frac{1}{\sqrt{(x-1)^2+2}}\,dx$

$= \left[\operatorname{ARSINH}\left(\frac{x-1}{\sqrt{2}}\right)\right]_1^2 = \operatorname{ARSINH}\left(\frac{1}{\sqrt{2}}\right) - \operatorname{ARSINH} 0$

$= \ln\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\right) - 0$

$= \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) = \ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$

b) $\int_0^1 e^{2x}\sinh x\,dx = \int_0^1 e^{2x}\left(\frac{1}{2}(e^x - e^{-x})\right)dx$

$= \frac{1}{2}\int_0^1 (e^{3x} - e^x)\,dx = \frac{1}{2}\left[\frac{1}{3}e^{3x} - e^x\right]_0^1$

$= \frac{1}{2}\left(\left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right)\right)$

$= \frac{1}{2}\left(\frac{1}{3}e^3 - e + \frac{2}{3}\right)$

$= \frac{1}{6}\left(e^3 - 3e + 2\right)$

Leave blank

4. Using the definitions of hyperbolic functions in terms of exponentials,

(a) show that

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

(3)

(b) solve the equation

$$4\sinh x - 3\cosh x = 3$$

(4)

a) $LHS = \operatorname{SECH}^2 x = \dfrac{1}{\cosh^2 x} = \left(\dfrac{2}{e^x + e^{-x}}\right)^2$

$RHS = 1 - \tanh^2 x = 1 - \dfrac{\sinh^2 x}{\cosh^2 x} = 1 - \dfrac{\left(\frac{1}{2}(e^x - e^{-x})\right)^2}{\left(\frac{1}{2}(e^x + e^{-x})\right)^2}$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2$$

HENCE LHS = RHS, AND SO $\operatorname{SECH}^2 x \equiv 1 - \tanh^2 x$,

AS REQUIRED.

//

Question 4 continued

b) $4\sinh x - 3\cosh x = 3$

$\Rightarrow 4\left(\frac{1}{2}(e^x - e^{-x})\right) - 3\left(\frac{1}{2}(e^x + e^{-x})\right) = 3$

$\Rightarrow (2e^x - 2e^{-x}) - \left(\frac{3}{2}e^x + \frac{3}{2}e^{-x}\right) = 3$

$\Rightarrow \frac{1}{2}e^x - \frac{7}{2}e^{-x} = 3$

$\Rightarrow e^x - 6 - 7e^{-x} = 0$

$\Rightarrow e^{2x} - 6e^x - 7 = 0$

$\Rightarrow (e^x - 7)(e^x + 1) = 0$

$\Rightarrow e^x = 7$ (SINCE e^x IS ALWAYS POSITIVE, e^x CANNOT EQUAL -1).

$\Rightarrow x = \ln 7$

5. Given that $y = \text{artanh} \dfrac{x}{\sqrt{(1+x^2)}}$

show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1+x^2)}}$ **(4)**

LET $u = \dfrac{x}{\sqrt{1+x^2}} \longrightarrow \cancel{\dfrac{du}{dx}}$

SO $y = \text{ARTANH } u$

$\Rightarrow \dfrac{dy}{du} = \dfrac{1}{1-u^2}$

$= \dfrac{1}{1-\left(\frac{x^2}{1+x^2}\right)}$

$= \dfrac{1+x^2}{(1+x^2)-x^2}$

$= 1+x^2$

$\dfrac{du}{dx} = \dfrac{\sqrt{1+x^2} - x\left(\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\right)(2x)}{1+x^2}$

$= \dfrac{\sqrt{1+x^2} - x^2(1+x^2)^{-\frac{1}{2}}}{1+x^2}$

$= \dfrac{(1+x^2) - x^2}{(1+x^2)^{\frac{3}{2}}}$

$= \dfrac{1}{(1+x^2)^{\frac{3}{2}}}$

SO $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = (1+x^2) \times \dfrac{1}{(1+x^2)^{\frac{3}{2}}}$

$= \dfrac{1}{(1+x^2)^{\frac{1}{2}}}$

$= \dfrac{1}{\sqrt{1+x^2}}$

AS REQUIRED

Leave blank

6. *[In this question you may use the appropriate trigonometric identities on page 6 of the pink Mathematical Formulae and Statistical Tables.]*

The points $P(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$, where $\alpha \neq \beta$, lie on the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

(a) Show the equation of the chord PQ is

$$\frac{x}{3}\cos\frac{(\alpha+\beta)}{2} + \frac{y}{2}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}$$

(4)

(b) Write down the coordinates of the mid-point of PQ.

(1)

Given that the gradient, m, of the chord PQ is a constant,

(c) show that the centre of the chord lies on a line

$$y = -kx$$

expressing k in terms of m.

(5)

a) GRADIENT OF $PQ = \dfrac{2\sin\alpha - 2\sin\beta}{3\cos\alpha - 3\cos\beta} = \dfrac{4\left(\cos\frac{\alpha+\beta}{2}\cancel{\sin\frac{\alpha-\beta}{2}}\right)}{-6\left(\sin\frac{\alpha+\beta}{2}\cancel{\sin\frac{\alpha-\beta}{2}}\right)}$

$$= \frac{-2\cos\frac{\alpha+\beta}{2}}{3\sin\frac{\alpha+\beta}{2}}$$

SO AN EQUATION OF CHORD PQ IS

$$y - 2\sin\alpha = \left(\frac{-2\cos\frac{\alpha+\beta}{2}}{3\sin\frac{\alpha+\beta}{2}}\right)(x - 3\cos\alpha)$$

$$\Rightarrow 3y\sin\frac{\alpha+\beta}{2} - 6\sin\alpha\sin\frac{\alpha+\beta}{2} = -2x\cos\frac{\alpha+\beta}{2} + 6\cos\alpha\cos\frac{\alpha+\beta}{2}$$

$$\Rightarrow 2x\cos\frac{\alpha+\beta}{2} + 3y\sin\frac{\alpha+\beta}{2} = 6\left(\cos\alpha\cos\frac{\alpha+\beta}{2} + \sin\alpha\sin\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \frac{x}{3}\cos\frac{\alpha+\beta}{2} + \frac{y}{2}\sin\frac{\alpha+\beta}{2} = \cos\left(\alpha - \left(\frac{\alpha+\beta}{2}\right)\right)$$

Leave blank

Question 6 continued

$$\Rightarrow \frac{x}{3}\cos\frac{\alpha+\beta}{2} + \frac{y}{2}\sin\frac{\alpha+\beta}{2} = \cos\left(\frac{\alpha-\beta}{2}\right)$$

AS REQUIRED.

b) MIDPOINT OF PQ HAS COORDINATES

$$\left(\frac{3}{2}(\cos\alpha+\cos\beta), (\sin\alpha+\sin\beta)\right)$$

c) GRADIENT OF PQ $= \dfrac{-2\cos\frac{\alpha+\beta}{2}}{3\sin\frac{\alpha+\beta}{2}} = m$

HENCE

$$-2\cos\frac{\alpha+\beta}{2} = 3m\sin\frac{\alpha+\beta}{2}.$$

USING TRIGONOMETRIC IDENTITIES, THE CENTRE OF CHORD'S COORDINATES CAN BE WRITTEN AS

$$\cancel{\left(3\cos\frac{\alpha+\beta}{2}, 2\sin\right.} \left(3\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}, 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)$$

ELIMINATING THE $\sin\frac{\alpha+\beta}{2}$ TERM,
THE y-COORDINATE CAN NOW BE WRITTEN AS

$$2\left(\frac{-2}{3m}\cos\frac{\alpha+\beta}{2}\right)\left(\cos\frac{\alpha-\beta}{2}\right) = \frac{-4}{3m}\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\text{SO } y = \frac{-4}{9m}\left(3\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow y = \frac{-4}{9m}x$$

Leave blank

7. A circle C with centre O and radius r has cartesian equation $x^2 + y^2 = r^2$ where r is a constant.

(a) Show that $1 + \left(\frac{dy}{dx}\right)^2 = \frac{r^2}{r^2 - x^2}$

(3)

(b) Show that the surface area of the sphere generated by rotating C through π radians about the x-axis is $4\pi r^2$.

(5)

(c) Write down the length of the arc of the curve $y = \sqrt{(1 - x^2)}$ from $x = 0$ to $x = 1$

(1)

a) DIFFERENTIATING IMPLICITLY, $2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

SQUARING GIVES $\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2}$

SO $1 + \left(\frac{dy}{dx}\right)^2 = \frac{y^2}{y^2} + \frac{x^2}{y^2} = \frac{x^2+y^2}{y^2} = \frac{r^2}{r^2-x^2}$

AS REQUIRED.

b) SURFACE AREA OF REVOLUTION $= 2\pi \int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= 2\pi \int_{-r}^{r} \sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\, dx$

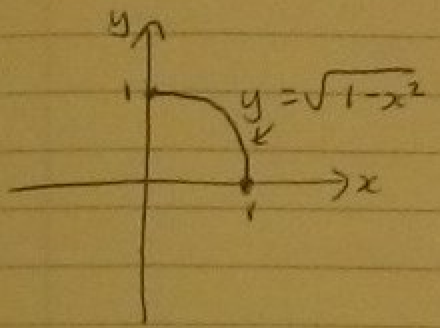
$= 2\pi \int_{-r}^{r} r\, dx \quad = 2\pi \left[rx\right]_{-r}^{r}$

$= 2\pi\left(r^2 - (-r^2)\right) = 4\pi r^2$, AS REQUIRED

Leave blank

Question 7 continued

c) THIS CURVE IS AN ARC OF A CIRCLE, CENTRE $(0,0)$ RADIUS 1.



HENCE THE ARC LENGTH IS $\frac{1}{4}\pi$

Leave blank

8. The position vectors of the points A, B and C from a fixed origin O are

$$\mathbf{a} = \mathbf{i} - \mathbf{j},\ \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k},\ \mathbf{c} = 2\mathbf{j} + \mathbf{k}$$

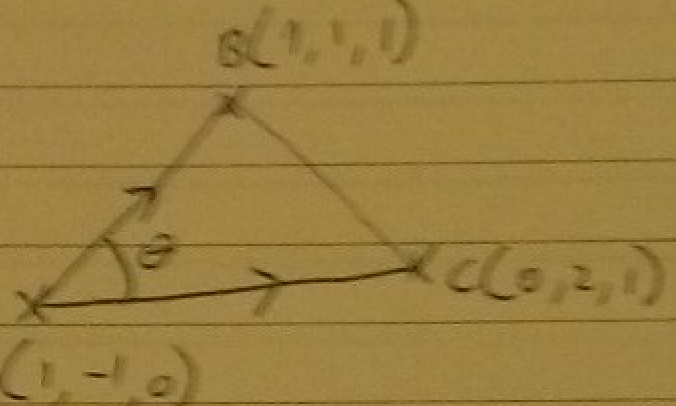
respectively.

(a) Using vector products, find the area of the triangle ABC. **(4)**

(b) Show that $\frac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0$ **(3)**

(c) Hence or otherwise, state what can be deduced about the vectors **a**, **b** and **c**. **(1)**

a)



AREA OF $\triangle ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

HENCE AREA OF $\triangle ABC = \frac{1}{2}\left|\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\right|$

$= \frac{1}{2}\left|\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\right| = \frac{1}{2}\sqrt{(-1)^2 + (-1)^2 + 2^2} = \frac{1}{2}\sqrt{6}$ //

b) $\frac{1}{6}\underline{a} . (\underline{b} \times \underline{c}) = \frac{1}{6}\begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \frac{1}{6}\big((1-2) - (-1-0) + 0\big)$

$= \frac{1}{6}\big(-1 - (-1) + 0\big) = 0$

AS REQUIRED //

c) VOLUME OF TETRAHEDRON OABC IS ZERO, HENCE VECTORS $\underline{a}$, $\underline{b}$ AND $\underline{c}$ ALL LIE IN (OR ARE PARALLEL TO) A SINGLE PLANE.

//

Leave blank

9. $$I_n = \int (x^2+1)^{-n}\,dx, \quad n > 0$$

(a) Show that, for $n > 0$

$$I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + \frac{2n-1}{2n} I_n$$

(5)

(b) Find I_2

(3)

a) $I_n = \int (x^2+1)^{-n}\,dx = \int (x^2+1)(x^2+1)^{-n-1}\,dx$

$\Rightarrow I_n = \int x^2(x^2+1)^{-n-1}\,dx + \int (x^2+1)^{-(n+1)}\,dx$

$\Rightarrow I_n = \int x^2(x^2+1)^{-n-1}\,dx + I_{n+1}$

$u = x \qquad v = \frac{-1}{2n}(x^2+1)^{-n}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = x(x^2+1)^{-n-1}$

$\Rightarrow I_n = \left(\frac{-1}{2n}x(x^2+1)^{-n} + \frac{1}{2n}\int (x^2+1)^{-n}\,dx\right) + I_{n+1}$

$\Rightarrow I_n = \frac{-1}{2n}x(x^2+1)^{-n} + \frac{1}{2n}I_n + I_{n+1}$

$\Rightarrow I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + I_n - \frac{1}{2n}I_n$

$\Rightarrow I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + \frac{2n-1}{2n}I_n$ AS REQUIRED

Leave blank

b) $I_1 = \int \frac{1}{x^2+1}\,dx = \text{ARCTAN}\,x \quad (+c)$

USING THE REDUCTION FORMULA,

$$I_2 = \frac{x(x^2+1)^{-1}}{2(1)} + \frac{2(1)-1}{2(1)} I_1$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\,\text{ARCTAN}\,x + C$$