FP2 UK S14

1. The complex numbers z_1 and z_2 are given by

$$z_1 = p + 2\text{i} \text{ and } z_2 = 1 - 2\text{i}$$

where p is an integer.

(a) Find $\frac{z_1}{z_2}$ in the form $a + b\text{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(4)

Given that $\left|\frac{z_1}{z_2}\right| = 13$,

(b) find the possible values of p.

(4)

a) $\frac{z_1}{z_2} = \frac{(p+2i) \times (1+2i)}{(1-2i) \times (1+2i)} = \frac{(p-4) + (2p+2)i}{1+4} = \frac{p-4}{5} + \left(\frac{2p+2}{5}\right)i$

b) $\left|\frac{z_1}{z_2}\right| = 13 \Rightarrow \frac{p^2+4}{1+4} = 169 \quad \therefore p^2+4 = 845$

$\therefore p^2 = 841 \quad \therefore p = \pm 29$

alt

$\left|\frac{z_1}{z_2}\right| = 13 \Rightarrow \left|\left(\frac{p-4}{5}\right) + \left(\frac{2p+2}{5}\right)i\right| = 13$

$\Rightarrow \left(\frac{p-4}{5}\right)^2 + \left(\frac{2p+2}{5}\right)^2 = 169 \quad = (p^2-8p+16) + (4p^2+8p+4) = 4225$

$\therefore 5p^2 + 20 = 4225$

$5p^2 = 4205$

$p^2 = 841 \quad \therefore p = \pm 29$

2.

$$f(x) = x^3 - \frac{5}{2x^{\frac{3}{2}}} + 2x - 3, \quad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[1.1, 1.5]$.

(2)

(b) Find $f'(x)$.

(2)

(c) Using $x_0 = 1.1$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(3)

a) $f(1.1) = -1.64$ $f(\alpha) = 0$ $\therefore$ $1.1 < \alpha < 1.5$
$f(1.5) = 2.01$ by sign change rule.

b) $f'(x) = 3x^2 + \frac{15}{4}x^{-\frac{5}{2}} + 2$

c) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.1 - \frac{(-1.64\ldots)}{f'(1.1)} = 1.29\underset{2}{1}$

3. Given that 2 and $1 - 5\text{i}$ are roots of the equation

$$x^3 + px^2 + 30x + q = 0, \quad p, q \in \mathbb{R}$$

(a) write down the third root of the equation. **(1)**

(b) Find the value of p and the value of q. **(5)**

(c) Show the three roots of this equation on a single Argand diagram. **(2)**

a) $z_1 = 2 \quad z_2 = 1 - 5i \quad z_3 = z_2^* = 1 + 5i$

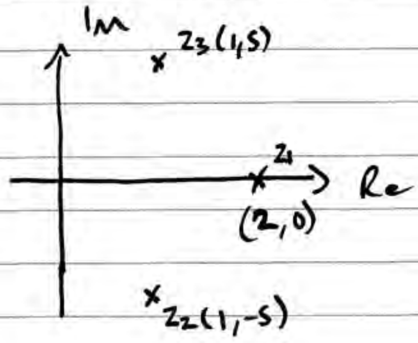
b) $(x-2)\left[(x-(1-5i))(x-(1+5i))\right] = 0$

$\Rightarrow (x-2)(x^2-2x+26) = 0$

$\Rightarrow$ (×x) $x^3 - 2x^2 + 26x$
(×−2) $-2x^2 + 4x - 52$ +

$x^3 - 4x^2 + 30x - 52 = 0 \therefore p = -4, \; q = -52$

c)



4. (i) Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 1 \end{pmatrix},$$

(a) find **AB**.

(b) Explain why $\mathbf{AB} \neq \mathbf{BA}$.

(4)

(ii) Given that

$$\mathbf{C} = \begin{pmatrix} 2k & -2 \\ 3 & k \end{pmatrix}, \text{ where } k \text{ is a real number}$$

find $\mathbf{C}^{-1}$, giving your answer in terms of k.

(3)

a) $AB = \begin{pmatrix} 4 & 5 & 6 \\ 5 & -6 & 11 \\ 13 & 11 & 21 \end{pmatrix}$

b) matrix multiplication is not commutative.

AB = 3x3 matrix BA = 3x2 matrix

ii) $\det C = 2k^2 + 6$

$\therefore C^{-1} = \dfrac{1}{2k^2+6} \begin{pmatrix} k & 2 \\ -3 & 2k \end{pmatrix}$

5. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(4n^2-1)$$

(6)

(b) Hence show that

$$\sum_{r=2n+1}^{4n}(2r-1)^2 = an(bn^2-1)$$

where a and b are constants to be found.

(3)

a $\sum_{1}^{n}(2r-1)^2 = 4\sum_{1}^{n} r^2 - 4\sum_{1}^{n} r + \sum_{1}^{n} 1$

$= 4 \times \frac{1}{6}n(n+1)(2n+1) - 4 \times \frac{1}{2}n(n+1) + n$

$= \frac{1}{3}n\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$

$= \frac{1}{3}n\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$

$= \frac{1}{3}n\left[4n^2 - 1\right]$ #

b) $\sum_{2n+1}^{4n} = \sum_{1}^{4n} - \sum_{1}^{2n} = \frac{1}{3}(4n)\left[4(4n)^2-1\right] - \frac{1}{3}(2n)\left[4(2n)^2-1\right]$

$= \frac{4}{3}n(64n^2-1) - \frac{2}{3}n(16n^2-1)$

$= \frac{2}{3}n\left[2(64n^2-1) - (16n^2-1)\right]$

$= \frac{2}{3}n\left[112n^2 - 1\right]$

6. The rectangular hyperbola H has cartesian equation $xy = c^2$.

The point $P\left(ct, \frac{c}{t}\right)$, $t > 0$, is a general point on H.

(a) Show that an equation of the tangent to H at the point P is

$$t^2 y + x = 2ct$$

(4)

An equation of the normal to H at the point P is $t^3x - ty = ct^4 - c$

Given that the normal to H at P meets the x-axis at the point A and the tangent to H at P meets the x-axis at the point B,

(b) find, in terms of c and t, the coordinates of A and the coordinates of B.

(2)

Given that $c = 4$,

(c) find, in terms of t, the area of the triangle APB. Give your answer in its simplest form.

(3)

a) $y = c^2x^{-1}$ $\quad \frac{dy}{dx} = -c^2x^{-2} = \frac{-c^2}{x^2}$

$P(ct, \frac{c}{t})$ $\quad M_t = \frac{-c^2}{c^2t^2} = -\frac{1}{t^2}$

$\therefore y - y_1 = m(x - x_1) \Rightarrow y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

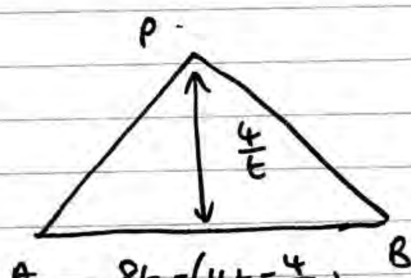
$\Rightarrow yt^2 - ct = -x + ct \quad \therefore t^2y + x = 2ct$

b) $y = 0 \Rightarrow t^3x = ct^4 - c$

$x = ct - \frac{c}{t^3} \Rightarrow A(ct - \frac{c}{t^3}, 0)$

$0 + x = 2ct \Rightarrow B(2ct, 0)$

c) $c = 4$ $\quad A(4t - \frac{4}{t^3}, 0)$ $\quad B(8t, 0)$ $\quad P(4t, \frac{4}{t})$



$8t - (4t - \frac{4}{t^3})$

$= 4t + \frac{4}{t^3}$

Area $= \left(2t + \frac{2}{t^3}\right)\left(\frac{4}{t}\right)$

$= 8 + \frac{8}{t^4}$

7. (i) In each of the following cases, find a 2×2 matrix that represents

(a) a reflection in the line $y = -x$,

(b) a rotation of 135° anticlockwise about (0, 0),

(c) a reflection in the line $y = -x$ followed by a rotation of 135° anticlockwise about (0, 0).

(4)

(ii) The triangle T has vertices at the points $(1, k)$, $(3, 0)$ and $(11, 0)$, where k is a constant.

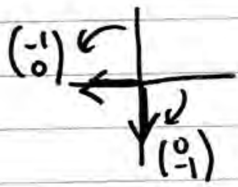
Triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}$$

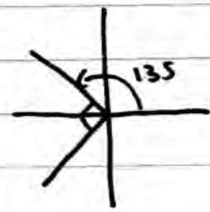
Given that the area of triangle T' is 364 square units, find the value of k.

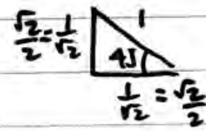
(6)

a) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



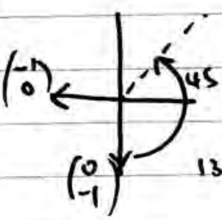
b)

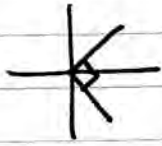




$\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

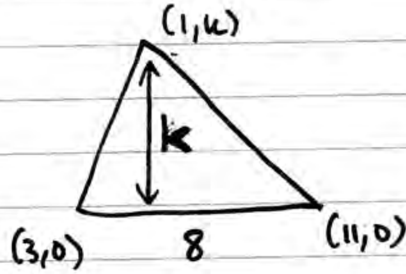
c)





$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$ alt $\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ =

ii) $\det(M) = 12 - (-2) = 14$ $\therefore$ area of $T = \frac{364}{14} = 26$



$\text{Area} = \frac{1}{2}(8) \times k = 4k = 26$

$\therefore k = 6.5$

8. The points $P(4k^2, 8k)$ and $Q(k^2, 4k)$, where k is a constant, lie on the parabola C with equation $y^2 = 16x$.

The straight line l_1 passes through the points P and Q.

(a) Show that an equation of the line l_1 is given by

$$3ky - 4x = 8k^2$$

(4)

The line l_2 is perpendicular to the line l_1 and passes through the focus of the parabola C. The line l_2 meets the directrix of C at the point R.

(b) Find, in terms of k, the y coordinate of the point R.

(7)

a) $m_{PQ} = \frac{4k}{3k^2} = \frac{4}{3k}$ $\quad (k^2, 4k)$

$y - y_1 = m(x - x_1)$ $\quad y - 4k = \frac{4}{3k}(x - k^2)$

$3ky - 12k^2 = 4x - 4k^2$

$\therefore 3ky - 4x = 8k^2$ #

b) $m_{l_2} = -\frac{3k}{4}$ $\quad y^2 = 4ax = 16x \quad \therefore a = 4$

$\therefore$ focus $(a, 0) = (4, 0)$

$y = -\frac{3k}{4}(x - 4) \Rightarrow 4y = 12k - 3kx$

directrix $x = -a$. $\quad \Rightarrow 4y = 12k + 3ka$

$\therefore 4y = 12k + 12k$

$4y = 24k$

$y = 6k$

9. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 8^n - 2^n$$

is divisible by 6

(6)

$n = 1 \quad f(1) = 8^1 - 2^1 = 8 - 2 = 6 = 6 \times 1 \quad \therefore$ true if $n = 1$

$n = 2 \quad f(2) = 8^2 - 2^2 = 64 - 4 = 60 = 6 \times 10 \quad \therefore$ true for $n = 2$

assume true for $n = k \quad \therefore \quad f(k) = 8^k - 2^k$ is divisible by 6

$$\begin{aligned} n = k+1 \quad f(k+1) &= 8^{k+1} - 2^{k+1} \\ &= 8(8^k) - 2(2^k) \\ &= 6(8^k) + 2(8^k) - 2(2^k) \\ &= 6(8^k) + 2[8^k - 2^k] \\ &= 6(8^k) + 2f(k) \quad \therefore \text{divisible by } 6. \end{aligned}$$

$\therefore$ true for $n = 1$, true for $n = k+1$ if true for $n = k$

$\therefore$ by Mathematical induction true for $n \in \mathbb{Z}^+$