Write your name here

Surname

Other names

Pearson Edexcel
International
Advanced Level

Centre Number

Candidate Number

Further Pure Mathematics F3

Advanced/Advanced Subsidiary

Tuesday 10 June 2014 – Morning
Time: 1 hour 30 minutes

Paper Reference
WFM03/01

You must have:
Mathematical Formulae and Statistical Tables (Blue)

Total Marks

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B). Coloured pencils and highlighter pens must not be used.
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- When a calculator is used, the answer should be given to an appropriate degree of accuracy.

Information

- The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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P 4 4 5 1 8 A 0 1 2 8

PEARSON

1. Given that $y = \arctan\left(\frac{2x}{3}\right)$,

(a) find $\frac{dy}{dx}$, giving your answer in its simplest form. **(2)**

(b) Use integration by parts to find

$$\int \arctan\left(\frac{2x}{3}\right) dx$$

(4)

a). $y = \arctan\left(\frac{2x}{3}\right)$ Let $u = \frac{2x}{3}$

$\frac{dy}{du} = \frac{1}{1+u^2}$ $\frac{du}{dx} = \frac{2}{3}$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{2x}{3}\right)}\left(\frac{2}{3}\right)$$

$$= \frac{2}{3\left(1+\frac{2x}{3}\right)}$$

$$= \frac{2}{3+2x}$$

b). u $\frac{du}{dx}$

$\arctan\left(\frac{2x}{3}\right)$ $\frac{2}{3+2x}$

v $\frac{dv}{dx}$

x 1

$$= x\arctan\left(\frac{2x}{3}\right) - \int \frac{2x}{3+2x}\,dx$$

$$= x\arctan\left(\frac{2x}{3}\right) - \left[\int \frac{2x+3-3}{2x+3}\,dx\right]$$

$$= x\arctan\left(\frac{2x}{3}\right) - \left[\int 1 - \frac{3}{2x+3}\,dx\right]$$

$$= x\arctan\frac{2x}{3} - x + \int \frac{3}{2x+3}\,dx$$

$$= x\arctan\left(\frac{2x}{3}\right) - x + \frac{3}{2}\ln(2x+3) + C$$

2. The line with equation $x = 9$ is a directrix of an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{8} = 1$$

where a is a positive constant.

Find the two possible exact values of the constant a.

(6)

$9 = \frac{a}{e}$ $\quad\quad$ $b^2 = 8$

$9e = a$

$a^2 = 81e^2$

$b^2 = a^2(1 - e^2)$

$8 = 81e^2(1 - e^2)$

$\frac{8}{81} = e^2 - e^4$

$\cancel{0 = e}$ $0 = e^4 - e^2 + \frac{8}{81}$

$e^2 = \frac{8}{9}$ or $e^2 = \frac{1}{9}$

$e = \frac{2\sqrt{3}}{3}$ or $e = \frac{1}{3}$

$9\left(\frac{2\sqrt{3}}{3}\right) = a$ or $\cancel{9e = \frac{1}{3}}$ $\quad 9\left(\frac{1}{3}\right) = a$

$a = 6\sqrt{3}$ $\quad\quad$ $\cancel{e = \frac{1}{27}}$ $\quad a = 3$

3. Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials,

(a) prove that

$$\cosh^2 x - \sinh^2 x = 1$$

(2)

(b) find algebraically the exact solutions of the equation

$$2 \sinh x + 7 \cosh x = 9$$

giving your answers as natural logarithms.

(5)

a). $LHS = \cosh^2 x - \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - [\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}]}{4}$$

$$= \frac{2+2}{4}$$

$$= 1 = RHS$$

b). $2\left(\frac{e^x - e^{-x}}{2}\right) + 7\left(\frac{e^x + e^{-x}}{2}\right) = 9$

$$2\cancel{4}(e^x - e^{-x}) + 7(e^x + e^{-x}) = 18$$

$$\cancel{11e^x + 3e^{-x} = 18}$$

$$\cancel{11e^{2x} - 18e^x + 3 = 0}$$

$$9e^x + 5e^{-x} - 18 = 0$$

$9e^{2x} - 18e^{x} + 5 = 0$

$e^{x} = \frac{5}{3}$ or $e^{x} = \frac{1}{3}$

$x = \ln\left(\frac{5}{3}\right)$ or $x = \ln\left(\frac{1}{3}\right)$

// //

blank

4. A non-singular matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 3 & k & 0 \\ k & 2 & 0 \\ k & 0 & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

(a) Find, in terms of k, the inverse of the matrix **M**.

(5)

The point A is mapped onto the point $(-5, 10, 7)$ by the transformation represented by the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(b) Find the coordinates of the point A.

(3)

a).

$$C = \begin{pmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} k & 0 \\ k & 1 \end{vmatrix} & \begin{vmatrix} k & 2 \\ k & 0 \end{vmatrix} \\ -\begin{vmatrix} k & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ k & 1 \end{vmatrix} & -\begin{vmatrix} 3 & k \\ k & 0 \end{vmatrix} \\ \begin{vmatrix} k & 0 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ k & 0 \end{vmatrix} & \begin{vmatrix} 3 & k \\ k & 2 \end{vmatrix} \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & -k & -2k \\ -k & 3 & +k^2 \\ 0 & -0 & 6-k^2 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 2 & -k & 0 \\ -k & 3 & 0 \\ -2k & k^2 & 6-k^2 \end{pmatrix}$$

$$\det(M) = 3(2)(1) + 0 + 0 - 0 - 0 - k^2$$

$$= 6 - k^2$$

$$M^{-1} = \frac{1}{6-k^2}\begin{pmatrix} 2 & -k & 0 \\ -k & 3 & 0 \\ -2k & k^2 & 6-k^2 \end{pmatrix}$$

b).

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} -5 \\ 10 \\ 7 \end{pmatrix}$$

$$= \frac{1}{6-1^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} -5 \\ 10 \\ 7 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -10 - 10 \\ 5 + 30 \\ 10 + 10 + 35 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -20 \\ 35 \\ 55 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 7 \\ 11 \end{pmatrix} //$$

A $(-4, 7, 11)$

5. Given that

$$I_n = \int_0^{\frac{\pi}{4}} \cos^n \theta \, d\theta, \qquad n \geqslant 0$$

(a) prove that, for $n \geqslant 2$,

$$nI_n = \left(\frac{1}{\sqrt{2}}\right)^n + (n-1)I_{n-2}$$

(6)

(b) Hence find the exact value of I_5, showing each step of your working.

(5)

a). $I_n = \int_0^{\frac{\pi}{4}} \cos^n\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} \cos^{n-1} \cos\theta \, d\theta$

u $\qquad$ $\frac{du}{d\theta}$

$\cos^{n-1}\theta$ $\qquad$ $(n-1)(\cos^{n-2})(-\sin\theta)$

v $\qquad$ $\frac{dv}{d\theta}$

$\sin\theta$ $\qquad$ $\cos\theta$

$= \left[\sin\theta \cos^{n-1}\theta\right]_0^{\frac{\pi}{4}} + (n-1)\int_0^{\frac{\pi}{4}} \cos^{n-2}\theta \sin^2\theta \, d\theta$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^{n-1} + (n-1)\int_0^{\frac{\pi}{4}} \cos^{n-2}\theta(1-\cos^2\theta)\, d\theta$

$I_n = \left(\frac{\sqrt{2}}{2}\right)^n + (n-1)\left[\int_0^{\frac{\pi}{4}} \cos^{n-2}\theta \, d\theta - \int_0^{\frac{\pi}{4}} \cos^n\theta \, d\theta\right]$

$I_n = \left(\frac{\sqrt{2}}{2}\right)^n + (n-1) I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = \left(\frac{\sqrt{2}}{2}\right)^n + (n-1)I_{n-2}$

$I_n = \left(\frac{\sqrt{2}}{2}\right)^n + (n-1)I_{n-1}$

b) n.

$$I_5 = \frac{\left(\frac{1}{\sqrt{2}}\right)^5 + (5-1)I_3}{5}$$

$$= \frac{\left(\frac{1}{\sqrt{2}}\right)^5 + 4I_3}{5}$$

$$= \frac{\frac{1}{4\sqrt{2}} + 4I_3}{5}$$

$$I_3 = \frac{\left(\frac{1}{\sqrt{2}}\right)^3 + (3-1)I_1}{3}$$

$$= \frac{\left(\frac{\sqrt{2}}{4}\right) + 2I_1}{3}$$

$$I_1 = \int_0^{\frac{\pi}{4}} \cos\theta \, d\theta$$

$$= \left[\sin\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{2}$$

$$I_3 = \frac{\frac{\sqrt{2}}{4} + \sqrt{2}}{3} = \frac{5\sqrt{2}}{12}$$

$$I_5 = \frac{\frac{\sqrt{2}}{8} + 4\left(\frac{5\sqrt{2}}{12}\right)}{5}$$

$$= \frac{43\sqrt{2}}{24} \div 5$$

$$= \frac{43\sqrt{2}}{120}$$

6. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

The line l is a tangent to H at the point P ($4\cosh\alpha$, $2\sinh\alpha$), where α is a constant, $\alpha \neq 0$

(a) Using calculus, show that an equation for l is

$$2y\sinh\alpha - x\cosh\alpha + 4 = 0$$

(4)

The line l cuts the y-axis at the point A.

(b) Find the coordinates of A in terms of α.

(2)

The point B has coordinates $(0, 10\sinh\alpha)$ and the point S is the focus of H for which $x > 0$

(c) Show that the line segment AS is perpendicular to the line segment BS. Bullshit

(5)

a). $x = 4\cosh\alpha \qquad y = 2\sinh\alpha$

$$\frac{dx}{d\alpha} = 4\sinh\alpha \qquad \frac{dy}{d\alpha} = 2\cosh\alpha$$

$$\frac{dy}{dx} = \frac{2\cosh\alpha}{4\sinh\alpha}$$

$$\frac{\cosh\alpha}{2\sinh\alpha} = \frac{y - 2\sinh\alpha}{x - 4\cosh\alpha}$$

$$x\cosh\alpha - 4\cosh^2\alpha = 2y\sinh\alpha - 4\sinh^2\alpha$$

$$0 = 2y\sinh\alpha - x\cosh\alpha + 4(\cosh^2\alpha - \sinh^2\alpha)$$

$$0 = 2y\sinh\alpha - x\cosh\alpha + 4(1)$$

b). $\because$ cut y axis

$\#6$

$\therefore \quad x=0$

$0 = 2y \sinh\alpha + 4$

$-4 = 2y \sinh\alpha$

$\cancel{-2\operatorname{csch}}\ -2\operatorname{csch}\alpha = y$ //

$(0, -2\operatorname{csch}\alpha)$

c). $\cancel{Si}$ $4 = 16(e^2-1)$

$\frac{4}{16} = e^2-1$

$e = \frac{\sqrt{5}}{2}$ //

Foci $(4(\frac{\sqrt{5}}{2}), 0)$

$= \cancel{S}(2\sqrt{5}, 0)$

$\cancel{M_A}\ m_{AS} = \frac{0 + 2\operatorname{csch}\alpha}{2\sqrt{5}}$ //

$M_{BS} = \frac{10\sinh\alpha - 0}{0 - 2\sqrt{5}}$

$= \frac{10\sinh\alpha}{-2\sqrt{5}} = -\frac{5\sinh\alpha}{\sqrt{5}}$

$= -\sqrt{5}\sinh\alpha$

$M_{AS} \cdot M_{BS} = \frac{\operatorname{csch}\alpha}{\sqrt{5}} \times -\sqrt{5}\sinh\alpha = -1$

$\therefore$ BS ~~Bullshit~~ ~~BS~~ $\perp$ AS.

7. The curve C has parametric equations

$$x = 3t^2, \quad y = 12t, \quad 0 \leqslant t \leqslant 4$$

The curve C is rotated through 2π radians about the x-axis.

(a) Show that the area of the surface generated is

$$\pi(a\sqrt{5} + b)$$

where a and b are constants to be found.

(6)

(b) Show that the length of the curve C is given by

$$k\int_0^4 \sqrt{(t^2 + 4)}\,dt$$

where k is a constant to be found.

(1)

(c) Use the substitution $t = 2\sinh\theta$ to show that the exact value of the length of the curve C is

$$24\sqrt{5} + 12\ln(2 + \sqrt{5})$$

(6)

a). $x = 3t^2 \qquad y = 12t$

$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 12$

$TSA = 2\pi\int_0^4 12t\sqrt{(6t)^2 + (12)^2}\,dt$

$= 2\pi\int_0^4 12t\sqrt{36t^2 + 144}\,dt$

$= 24\pi\int_0^4 t\sqrt{36}\sqrt{t^2+4}\,dt$

$= 144\pi\int_0^4 t\sqrt{t^2+4}\,dt$

$= \frac{144}{3}\pi\left[(t^2+4)^{\frac{3}{2}}\right]_0^4$

$(t^2+4)^{\frac{3}{2}}$

$= \frac{3}{2}(t^2+4)^{\frac{1}{2}}2t$

$= 3(t^2+4)^{\frac{1}{2}}t$

$$= 48\pi \left[(4^2+4)^{\frac{3}{2}} - (4)^{\frac{3}{2}} \right]$$

$$= 48\pi \left[40\sqrt{5} - 8 \right]$$

$$= \pi \left[1920\sqrt{5} - 384 \right]$$

$a = 1920, \quad b = -384$

b). $S = \int_0^4 \sqrt{(t^2+4)}\ \sqrt{36}\, dt$

$$= 6 \int_0^4 \sqrt{t^2+4}\ dt$$

$k = 6$

c). Limits:

$4 = 2\sinh\theta$ $\qquad$ $0 = 2\sinh\theta$

$2 = \sinh\theta$ $\qquad$ $0 = \sinh\theta$

$\theta = \ln(2+\sqrt{5})$ $\qquad$ $\theta = 0$

$$t = 2\sinh\theta$$

$$dt = 2\cosh\theta\, d\theta$$

$$S = 6 \int_0^{\cancel{4}\ \ln(2+\sqrt{5})} \sqrt{4\sinh^2\theta + 4}\quad 2\cosh\theta\, d\theta$$

$$= 12 \int_0^{\ln(2+\sqrt{5})} \sqrt{4}\cosh^2\theta\, d\theta$$

$$= 6 \int_0^{\ln(2+\sqrt{5})} 2(\cosh 2\theta + 1)\, d\theta$$

$$= \cancel{6}\,12 \left[\tfrac{1}{2}\sinh 2\theta + \theta \right]_0^{\ln(2+\sqrt{5})}$$

$$= \cancel{6\int} \overset{12}{\cancel{6}} \left[\frac{1}{2} \sinh 2\ln(2+\sqrt{5}) + \ln(2+\sqrt{5}) \right] + 0$$

$$= 6\cancel{3} \sinh \ln(2+\sqrt{5})^2 + 12\ln(2+\sqrt{5}) + 0$$

$$= \cancel{3}^{6} \left(\frac{e^{\ln(2+\sqrt{5})^2} - e^{-\ln(2+\sqrt{5})^2}}{2} \right) + 12\ln(2+\sqrt{5})$$

$$= \overset{\cancel{3}}{\cancel{6}}3 \left((2+\sqrt{5})^2 \cancel{+} - (2+\sqrt{5})^{-2} \right) \cancel{+\ln(2+\sqrt{5})} + 12\ln(2+\sqrt{5})$$

$$= 3\left(9+4\sqrt{5} - (9-4\sqrt{5}) \right) + 12\ln(2+\sqrt{5})$$

$$= 3(8\sqrt{5}) + 12\ln(2+\sqrt{5})$$

$$= 24\sqrt{5} + 12\ln(2+\sqrt{5})$$

8. The line l has equation

$$\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \lambda(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \text{ where } \lambda \text{ is a scalar parameter,}$$

and the plane Π has equation

$$\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 19$$

(a) Find the coordinates of the point of intersection of l and Π. **(4)**

The perpendicular to Π from the point A (2, 1, –2) meets Π at the point B.

(b) Verify that the coordinates of B are (4, 3, –6). **(3)**

The point A (2, 1, –2) is reflected in the plane Π to give the image point A'.

(c) Find the coordinates of the point A'. **(2)**

(d) Find an equation for the line obtained by reflecting the line l in the plane Π, giving your answer in the form

$$\mathbf{r} \times \mathbf{a} = \mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors to be found. **(4)**

a, $r = \begin{pmatrix} 2 + 3\lambda \\ 1 + 2\lambda \\ 2 + \lambda \end{pmatrix}$

$$\begin{pmatrix} 2+3\lambda \\ 1+2\lambda \\ 2+\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 19$$

$$2 + 3\lambda + 1 + 2\lambda - 4 - 4\lambda = 19$$

$$\lambda - 1 = 19$$

$$\lambda = 20$$

$\therefore$ point of intersection is

$$\begin{pmatrix} 2 + 3(20) \\ 1 + 2(20) \\ 2 + 20 \end{pmatrix} = \begin{pmatrix} 62 \\ 23 \\ 22 \end{pmatrix}$$

d)

$$a = \begin{pmatrix} 6 \\ 4 \\ -10 \end{pmatrix}$$