1.

$$f(x) = 2x^3 + x - 10$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[1.5, 2]$

(2)

The only real root of $f(x) = 0$ is α

The iterative formula

$$x_{n+1} = \left(5 - \frac{1}{2}x_n\right)^{\frac{1}{3}}, \quad x_0 = 1.5$$

can be used to find an approximate value for α

(b) Calculate x_1, x_2 and x_3, giving your answers to 4 decimal places.

(3)

(c) By choosing a suitable interval, show that $\alpha = 1.6126$ correct to 4 decimal places.

(2)

a) $f(1.5) = -1.75 < 0$ $\quad f(2) = 8 > 0$ $\quad \therefore$ by sign change rule $1.5 < \alpha < 2$

b) $x_0 = 1.5 \quad x_1 = 1.6198 \quad x_2 = 1.6122 \quad x_3 = 1.6126$

c) $f(1.61255) = -0.001166 < 0$ $\quad \therefore$ by sign change rule
$f(1.61265) = 0.000493 > 0$

$\alpha = 1.6126$

2. A curve C has the equation

$$x^3 - 3xy - x + y^3 - 11 = 0$$

Find an equation of the tangent to C at the point $(2, -1)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(6)

$$\frac{d}{dx}(x^3 - 3xy - x + y^3 - 11)$$

$$3x^2 - 3x\frac{dy}{dx} - 3y - 1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (3y^2 - 3x)\frac{dy}{dx} = 1 + 3y - 3x^2$$

$$\therefore \frac{dy}{dx} = \frac{1 + 3y - 3x^2}{3y^2 - 3x} \quad \text{at } (2,-1)$$

$$M_t = \frac{1 - 3 - 12}{3 - 6} = \frac{14}{3}$$

$$y + 1 = \frac{14}{3}(x - 2) \Rightarrow 3y + 3 = 14x - 28$$

$$14x - 3y - 31 = 0$$

3. Given that

$$y = \frac{\cos 2\theta}{1 + \sin 2\theta}, \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

show that

$$\frac{dy}{d\theta} = \frac{a}{1 + \sin 2\theta}, \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

where a is a constant to be determined.

(4)

$u = \cos 2\theta \qquad v = 1 + \sin 2\theta$

$u' = -2\sin 2\theta \qquad v' = 2\cos 2\theta$

$$\therefore \frac{dy}{d\theta} = \frac{-2\sin 2\theta(1+\sin 2\theta) - 2\cos^2 2\theta}{(1+\sin 2\theta)^2}$$

$$= \frac{-2\sin 2\theta - 2\overbrace{(\sin^2 2\theta + \cos^2 2\theta)}^{=1}}{(1+\sin 2\theta)^2}$$

$$= \frac{-2(1+\sin 2\theta)}{(1+\sin 2\theta)^2} = \frac{-2}{1+\sin 2\theta}$$

4. Find

(a) $\int (2x+3)^{12}\,\mathrm{d}x$

(2)

(b) $\int \frac{5x}{4x^2+1}\,\mathrm{d}x$

(2)

a) $\frac{1}{13}(2x+3)^{13} \div 2 = \frac{1}{26}(2x+3)^{13} + c$

b) $\frac{5}{8}\int \frac{8x}{4x^2+1}\,dx = \frac{5}{8}\ln(4x^2+1) + c$

5. $$f(x) = (8 + 27x^3)^{\frac{1}{3}}, \quad |x| < \frac{2}{3}$$

Find the first three non-zero terms of the binomial expansion of f(x) in ascending powers of x. Give each coefficient as a simplified fraction.

(5)

$$8^{\frac{1}{3}}\left(1 + \frac{27}{8}x^3\right)^{\frac{1}{3}}$$

$$= 2\left[1 + \left(\frac{1}{3}\right)\left(\frac{27}{8}x^3\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(\frac{27}{8}x^3\right)^2\right]$$

$$= 2 + \frac{9}{4}x^3 + -\frac{81}{32}x^6$$

6. (a) Express $\dfrac{5-4x}{(2x-1)(x+1)}$ in partial fractions.

(3)

(b) (i) Find a general solution of the differential equation

$$(2x-1)(x+1)\frac{dy}{dx} = (5-4x)y, \quad x > \frac{1}{2}$$

Given that $y = 4$ when $x = 2$,

(ii) find the particular solution of this differential equation.
Give your answer in the form $y = f(x)$.

(7)

$5 x^4$ $\quad \dfrac{5-4x}{(2x-1)(x+1)} = \dfrac{A}{2x-1} + \dfrac{B}{x+1}$

$5-4x = A(x+1) + B(2x-1)$

$x = -1 \Rightarrow 9 = -3B \quad \therefore B = \frac{-3}{2}$ $\qquad \therefore \dfrac{2}{2x-1} - \dfrac{3}{x+1}$

$x = \frac{1}{2} \Rightarrow 3 = 1\frac{1}{2}A \quad A = \frac{2}{2}$

2

b) $\displaystyle\int \frac{1}{y}\,dy = \int \frac{(5-4x)}{(2x-1)(x+1)}\,dx$

$\therefore \ln y = \ln(2x-1) - 3\ln(x+1) + c$

$(2,4)$ $\quad \ln 4 = \ln 3 - 3\ln 3 + c \qquad \ln 4 = -2\ln 3 + c$

$c = \ln 4 + 2\ln 3 = \ln 4 + \ln 9 = \ln 36.$

$\therefore \ln y = \ln(2x-1) - \ln(x+1)^3 + \ln 36$

$\therefore y = \dfrac{36(2x-1)}{(x+1)^3}$

7. The function f is defined by

$$f : x \mapsto \frac{3x-5}{x+1}, \quad x \in \mathbb{R}, x \neq -1$$

(a) Find an expression for $f^{-1}(x)$

(3)

(b) Show that

$$ff(x) = \frac{x+a}{x-1}, \quad x \in \mathbb{R}, x \neq -1, x \neq 1$$

where a is an integer to be determined.

(4)

The function g is defined by

$$g : x \mapsto x^2 - 3x, \quad x \in \mathbb{R}, 0 \leqslant x \leqslant 5$$

(c) Find the value of fg(2)

(2)

(d) Find the range of g

(3)

a) $x = \frac{3y-5}{y+1} \Rightarrow xy + x = 3y - 5 \Rightarrow 3y - xy = x + 5$

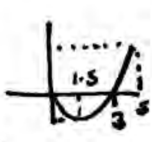
$y(3-x) = x+5 \quad \therefore \quad y = \frac{x+5}{3-x}$ #

b) $ff(x) = \frac{3\left(\frac{3x-5}{x+1}\right) - 5}{\left(\frac{3x-5}{x+1}\right)+1} = \frac{\frac{9x-15-5(x+1)}{(\cancel{x+1})}}{\frac{3x-5+(x+1)}{(\cancel{x+1})}}$

$= \frac{4x-20}{4x-4} = \frac{\cancel{4}(x-5)}{\cancel{4}(x-1)} = \frac{x-5}{x-1}$

c) $fg(2) = f(2^2 - 3(2)) = f(-2) = \frac{-6-5}{-2+1} = \frac{-11}{-1} = 11$

d) $g(x) = x(x-3)$



$g(1.5) = -\frac{9}{4}$

$g(5) = 10$

$-\frac{9}{4} \leq g(x) \leq 10$

8. The volume V of a spherical balloon is increasing at a constant rate of $250\ \text{cm}^3\ \text{s}^{-1}$. Find the rate of increase of the radius of the balloon, in cm s^{-1}, at the instant when the volume of the balloon is $12\,000\ \text{cm}^3$.
Give your answer to 2 significant figures.

(5)

[You may assume that the volume V of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$.]

$\frac{dV}{dt} = 250$ find $\frac{dr}{dt}$ when $V = 12000$

$\frac{dV}{dr} = 4\pi r^2$ $\qquad \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$\frac{dr}{dt} = \left(\frac{1}{4\pi r^2}\right)(250) = \frac{250}{4\pi r^2}$ $\qquad V = 12000 = \frac{4}{3}\pi r^3$

$r = 14.2...$

$= 0.099$ cm/sec

9.

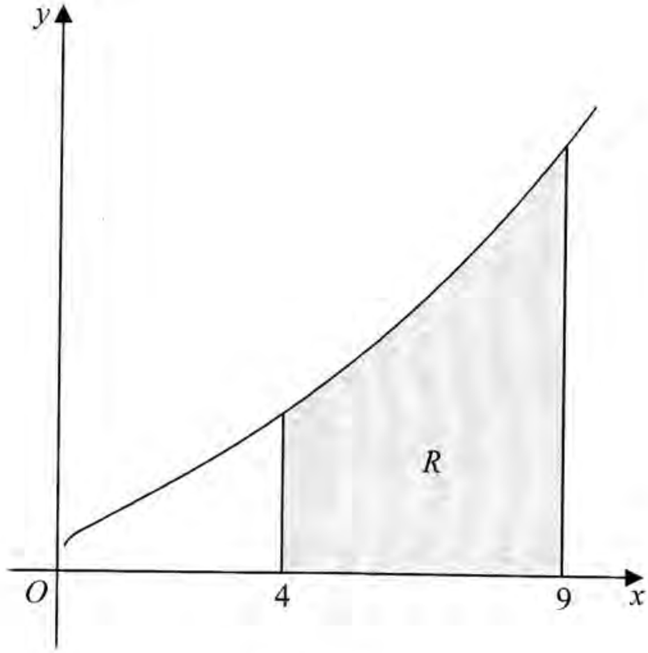


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = e^{\sqrt{x}}$, $x > 0$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the lines $x = 4$ and $x = 9$

(a) Use the trapezium rule, with 5 strips of equal width, to obtain an estimate for the area of R, giving your answer to 2 decimal places.

(4)

(b) Use the substitution $u = \sqrt{x}$ to find, by integrating, the exact value for the area of R.

(7)

$h=1$

x	4	5	6	7	8	9
y	e^2	$e^{\sqrt{5}}$	$e^{\sqrt{6}}$	$e^{\sqrt{7}}$	$e^{\sqrt{8}}$	e^3

Area $\approx \frac{1}{2}(e^2+2(e^{\sqrt{5}}+\dots)+e^3)$

≈ 65.69

2

b) $u = x^{\frac{1}{2}}$ $\quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad dx = 2x^{\frac{1}{2}}du = 2u\,du$

$\therefore \int_4^9 y\,dx = \int_2^3 e^u \times 2u\,du$ $\quad u = 2u \quad v = e^u$

$u' = 2 \quad v' = e^u$

$= \left[2ue^u - 2\int e^u du\right]_2^3 = \left[2ue^u - 2e^u\right]_2^3$

$= \left[2e^u(u-1)\right]_2^3 = 4e^3 - 2e^2$

2

10. (a) Use the identity for $\sin(A + B)$ to prove that

$$\sin 2A \equiv 2\sin A \cos A$$

(2)

(b) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(\tan\left(\tfrac{1}{2}x\right)\right)\right] = \operatorname{cosec} x$$

(4)

A curve C has the equation

$$y = \ln\left(\tan\left(\tfrac{1}{2}x\right)\right) - 3\sin x, \qquad 0 < x < \pi$$

(c) Find the x coordinates of the points on C where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

Give your answers to 3 decimal places.
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

a) let $A=B$ $\quad \sin(A+B) = \sin A\cos B + \cos A \sin B$

$\sin(A+A) = \sin A\cos A + \cos A\sin A$

$\sin 2A \equiv 2\sin A\cos A$ #

b) $\frac{d}{dx}\left(\ln\left[\tan\left(\tfrac{1}{2}x\right)\right]\right) = \frac{\frac{1}{2}\sec^2\left(\frac{1}{2}x\right)}{\tan\left(\frac{1}{2}x\right)} = \frac{1}{2\left[\cos\left(\frac{1}{2}x\right)\right]^2} \times \frac{\cos\left(\frac{1}{2}x\right)}{\sin\left(\frac{1}{2}x\right)}$

$= \frac{1}{2\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)} = \frac{1}{\sin x} = \operatorname{cosec} x$ #

c) $\frac{dy}{dx} = 0 \Rightarrow \operatorname{cosec} x - 3\cos x = 0 \Rightarrow 3\cos x = \frac{1}{\sin x}$

$\Rightarrow 3\sin x\cos x = 1 \Rightarrow 2\sin x\cos x = \frac{2}{3}$

$\Rightarrow \sin 2x = \frac{2}{3} \Rightarrow 2x = \sin^{-1}\left(\frac{2}{3}\right) = 0.7297\ldots;\ 2.412$

$\therefore x = 0.3648\ldots$

$x = 0.365^c$

$x = 1.206^c$

11.

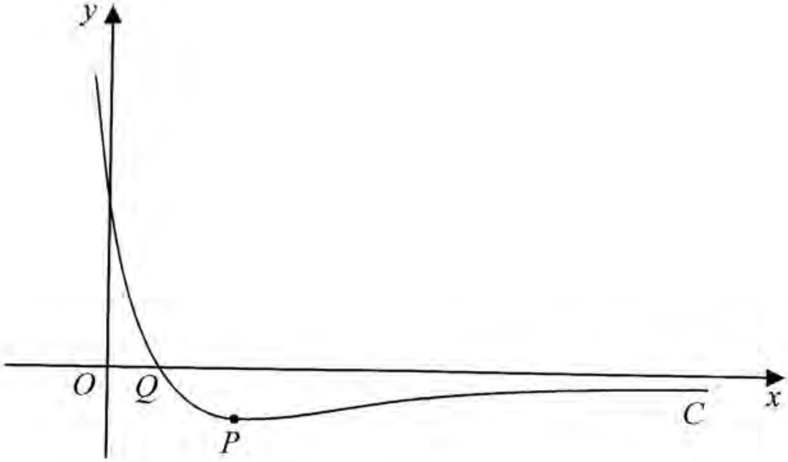


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = e^{a-3x} - 3e^{-x}, \quad x \in \mathbb{R}$$

where a is a constant and $a > \ln 4$

The curve C has a turning point P and crosses the x-axis at the point Q as shown in Figure 2.

(a) Find, in terms of a, the coordinates of the point P.

(6)

(b) Find, in terms of a, the x coordinate of the point Q.

(3)

(c) Sketch the curve with equation

$$y = \left|e^{a-3x} - 3e^{-x}\right|, \quad x \in \mathbb{R}, \ a > \ln 4$$

Show on your sketch the exact coordinates, in terms of a, of the points at which the curve meets or cuts the coordinate axes.

(3)

$\frac{dy}{dx} = -3e^{a-3x} + 3e^{-x} = 0 \qquad 3e^{-x} = 3e^{a-3x}$

$\therefore -x = a - 3x$

$2x = a$

$x = \frac{1}{2}a$

$y = e^{a-\frac{3}{2}a} - 3e^{-\frac{a}{2}} = e^{-\frac{a}{2}} - 3e^{-\frac{a}{2}}$

$\left(\frac{1}{2}a, -2e^{-\frac{a}{2}}\right)$

b) $y=0 \Rightarrow e^{a-3x} = 3e^{-x}$

$\ln e^{a-3x} = \ln 3e^{-x}$

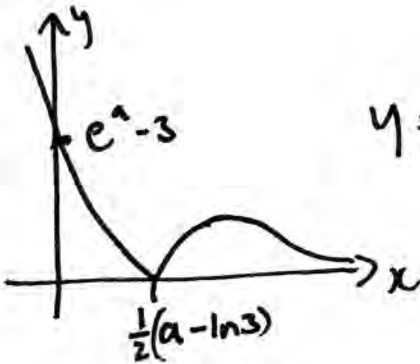
$a-3x = \ln 3 + \ln e^{-x}$

$a-3x = \ln 3 - x \qquad \therefore 2x = a - \ln 3$

$x = \frac{1}{2}(a-\ln 3)$

2

c)



$y = |e^{a-3x} - 3e^{-x}|$

$x=0 \quad y = e^a - 3$

12.

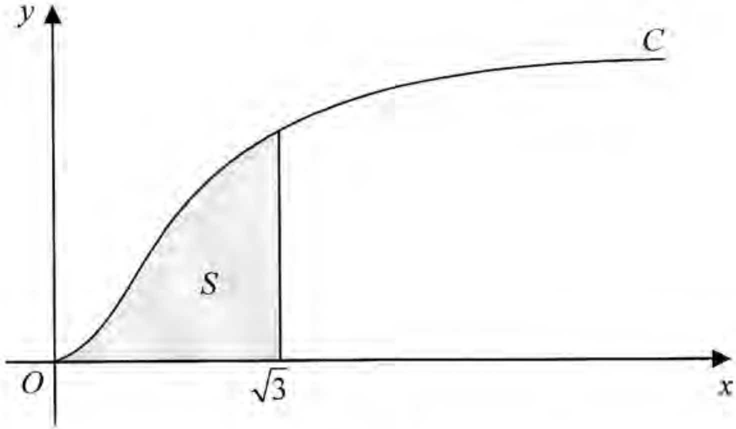


Figure 3

Figure 3 shows a sketch of part of the curve C with parametric equations

$$x = \tan t, \quad y = 2\sin^2 t, \quad 0 \leqslant t < \frac{\pi}{2}$$

The finite region S, shown shaded in Figure 3, is bounded by the curve C, the line $x = \sqrt{3}$ and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid of revolution formed is given by

$$4\pi \int_0^{\frac{\pi}{3}} (\tan^2 t - \sin^2 t)\,\mathrm{d}t$$

(6)

(b) Hence use integration to find the exact value for this volume.

(6)

12a) $\text{Volume} = \pi \int y^2 dx = \pi \int_{x=0}^{x=\sqrt{3}} y^2 \frac{dx}{dt} dt$ $\tan t = \sqrt{3} \therefore t = \frac{\pi}{3}$

$\tan t = 0 \therefore t = 0$

$x = \tan t \qquad y^2 = 4\sin^4 t$

$\frac{dx}{dt} = \sec^2 t \qquad \therefore \text{Volume} = 4\pi \int_0^{\frac{\pi}{3}} \sin^4 t \sec^2 t \, dt$

$= 4\pi \int_0^{\frac{\pi}{3}} \frac{\sin^2 t(1-\cos^2 t)}{\cos^2 t} dt \, t = 4\pi \int_0^{\frac{\pi}{3}} \tan^2 t(1-\cos^2 t) dt$

$= 4\pi \int_0^{\pi/3} \tan^2 t - \frac{\sin^2 t}{\cos^2 t} \times \cos^2 t \, dt = 4\pi \int_0^{\pi/3} \tan^2 t - \sin^2 t \, dt$

b) $\frac{\sin^2}{\cos^2} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2} \Rightarrow \tan^2 + 1 = \sec^2$ $\qquad \cos 2t = 1 - 2\sin^2 t$

$-1 + \sec^2 x = \tan^2 x \qquad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$

$= 4\pi \int_0^{\frac{\pi}{3}} \sec^2 x - 1 - \frac{1}{2} + \frac{1}{2}\cos 2t \, dt$

$= 4\pi \int_0^{\frac{\pi}{3}} \sec^2 x + \frac{1}{2}\cos 2t - \frac{3}{2} dt = 2\pi \int_0^{\frac{\pi}{3}} 2\sec^2 x + \cos 2t - 3 \, dt$

$= 2\pi \left[2\tan x + \frac{1}{2}\sin 2t - 3t \right]_0^{\frac{\pi}{3}}$

$= 2\pi \left[\left(2\sqrt{3} + \frac{\sqrt{3}}{4} - \pi\right) - (0) \right]$

$= 2\pi \left[\frac{9\sqrt{3}}{4} - \pi \right] = \frac{1}{2}\pi\left(9\sqrt{3} - 4\pi\right)$

13. (a) Express $2\sin\theta + \cos\theta$ in the form $R\sin(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Give your value of α to 2 decimal places. **(3)**

$R\sin(\theta+\alpha) = R\cancel{\sin\theta}\cos\alpha + R\cancel{\cos\theta}\sin\alpha$ $\quad$ $R\sin\alpha = 1$

$2\cancel{\sin\theta} \quad\quad +1\cancel{\cos\theta}$ $\quad$ $R\cos\alpha = 2$

$\tan\alpha = \frac{1}{2}$ $\alpha = 0.463647$ $\alpha = 0.46^c$ $R^2 = 1^2 + 2^2 \Rightarrow R = \sqrt{5}$

$\sqrt{5}\sin(\theta + 0.46^c) \simeq \sqrt{5}\sin(\theta + 26.57°)$

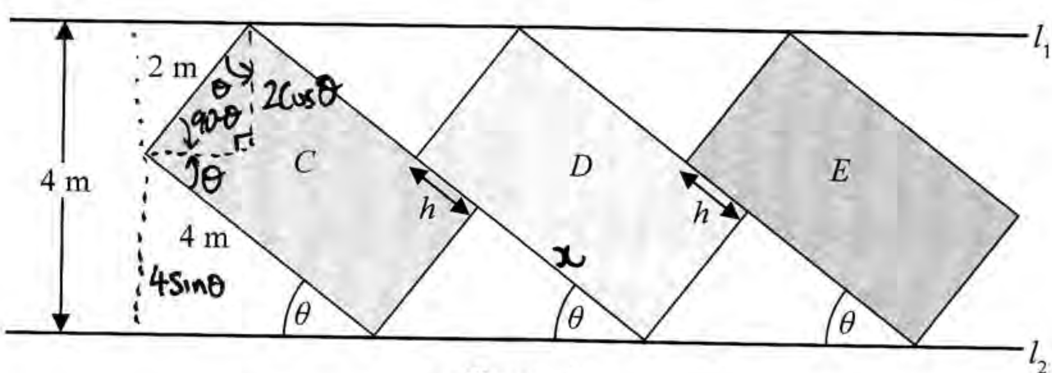


Figure 4

Figure 4 shows the design for a logo that is to be displayed on the side of a large building. The logo consists of three rectangles, C, D and E, each of which is in contact with two horizontal parallel lines l_1 and l_2. Rectangle D touches rectangles C and E as shown in Figure 4.

Rectangles C, D and E each have length 4 m and width 2 m. The acute angle θ between the line l_2 and the longer edge of each rectangle is shown in Figure 4.

Given that l_1 and l_2 are 4 m apart,

(b) show that

$$2\sin\theta + \cos\theta = 2$$

(2)

Given also that $0 < \theta < 45°$,

(c) solve the equation

$$2\sin\theta + \cos\theta = 2$$

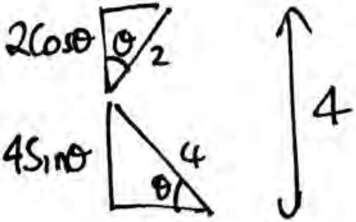
giving the value of θ to 1 decimal place. **(3)**

Rectangles C and D and rectangles D and E touch for a distance h m as shown in Figure 4.

Using your answer to part (c), or otherwise,

(d) find the value of h, giving your answer to 2 significant figures. **(3)**

b)



$\therefore\ 4\sin\theta + 2\cos\theta = 4$

$\therefore\ 2\sin\theta + \cos\theta = 2$

c) $\sqrt{5}\sin(\theta + 0.469) = 2$

$\theta + 26.57 = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) = 63.4,\ 116.56\ldots$

$\therefore\ \theta = 36.9°$

d)

$\tan\theta = \frac{2}{x}\ \therefore\ x = \frac{2}{\tan\theta} = \frac{8}{3}$

$\therefore\ h = 4 - \frac{8}{3} = \frac{4}{3}\text{m} = 1.3\text{m}$

14. Relative to a fixed origin O, the line l has vector equation

$$\mathbf{r} = \begin{pmatrix} -1 \\ -4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

where λ is a scalar parameter.

Points A and B lie on the line l, where A has coordinates $(1, a, 5)$ and B has coordinates $(b, -1, 3)$.

(a) Find the value of the constant a and the value of the constant b.

(3)

(b) Find the vector $\overrightarrow{AB}$.

(2)

The point C has coordinates $(4, -3, 2)$

(c) Show that the size of the angle CAB is $30°$

(3)

(d) Find the exact area of the triangle CAB, giving your answer in the form $k\sqrt{3}$, where k is a constant to be determined.

(2)

The point D lies on the line l so that the area of the triangle CAD is twice the area of the triangle CAB.

(e) Find the coordinates of the two possible positions of D.

(4)

a) $\begin{pmatrix} -1+2\lambda \\ -4+\lambda \\ 6-\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ a \\ 5 \end{pmatrix}$ $\lambda = 1$ $\therefore$ $\underline{a = -3}$ $\begin{pmatrix} -1+2\lambda \\ -4+\lambda \\ 6-\lambda \end{pmatrix} = \begin{pmatrix} b \\ -1 \\ 3 \end{pmatrix}$ $\lambda = 3$ $\therefore \underline{b = 5}$

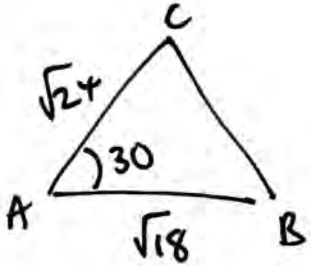
b) $\overrightarrow{AB} = b - a = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

c) $\overrightarrow{AC} = c - a = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

$\cos\theta = \dfrac{\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}}{\left|\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}\right|} \Rightarrow \cos\theta = \left(\dfrac{18}{\sqrt{24 \times 18}}\right) \therefore \cos\theta = \dfrac{18}{12\sqrt{3}}$

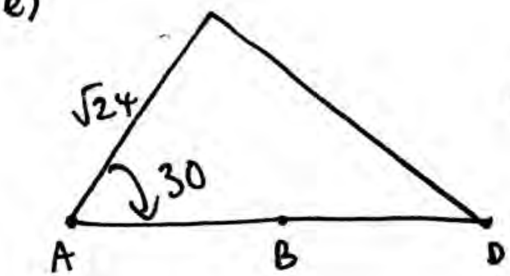
$\therefore \cos\theta = \dfrac{18\sqrt{3}}{36} = \dfrac{\sqrt{3}}{2} \therefore \underline{\theta = 30°}$

d)



Area $= \frac{1}{2}\sqrt{24}\sqrt{18}\,\sin 30 = \frac{12\sqrt{3}}{4} = \frac{3\sqrt{3}}{3}$

e)



Area $= 6\sqrt{3}$

$\frac{1}{2}\sqrt{24}\,|\vec{AD}|\,\sin 30 = 6\sqrt{3}$

$\therefore |\vec{AD}| = \frac{24\sqrt{3}}{\sqrt{24}} = 6\sqrt{2}$

$AB = \sqrt{18} = 3\sqrt{2}$.

$\therefore \vec{AD} = 2 \times \vec{AB}$

$d = a \pm 2\vec{AB}$

$d = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \pm 2\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -7 \\ -7 \\ 9 \end{pmatrix}$

$(9,1,1)$ or $(-7,7,9)$