FP2 S14 (R)

1. The roots of the equation

$$2z^3 - 3z^2 + 8z + 5 = 0$$

are z_1, z_2 and z_3

Given that $z_1 = 1 + 2i$, find z_2 and z_3

(5)

$z_2 = 1-2i$

$(2z-\alpha)(z-(1+2i))(z-(1-2i))=0$

$(2z-\alpha)(z^2-2z+5)=0$ $\quad -5\alpha \equiv 5 \therefore \alpha = -1$

$(2z+1)=0 \therefore z_3 = -\frac{1}{2}$ $\quad \underline{\underline{z_1 = 1+2i}} \; \underline{\underline{z_2 = 1-2i}} \; \underline{\underline{z_3 = -\frac{1}{2}}}$

2.

$$f(x) = 3\cos 2x + x - 2, \quad -\pi \leqslant x < \pi$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [2, 3].

(2)

(b) Use linear interpolation once on the interval [2, 3] to find an approximation to α.

Give your answer to 3 decimal places.

(3)

(c) The equation $f(x) = 0$ has another root β in the interval [–1, 0]. Starting with this interval, use interval bisection to find an interval of width 0.25 which contains β.

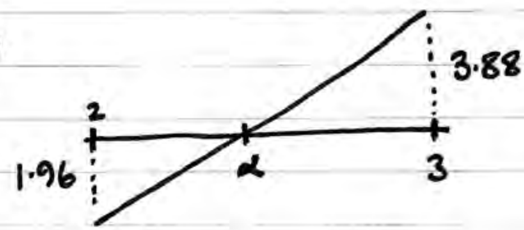
(4)

a) $f(2) = -1.96$ $\quad f(x) = 0$ So by sign change rule

$f(3) = 3.88$

$2 < \alpha < 3$

b)



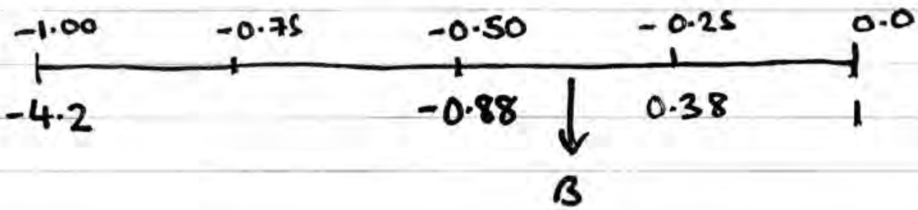
$$\frac{\alpha - 2}{1.96} = \frac{3 - \alpha}{3.88}$$

$3.88\alpha - 7.761.. = 5.882.. - 1.96\alpha$

$\therefore 5.84144..\alpha = 13.6438...$

$\therefore \alpha = 2.336$

c)



$\therefore$

$-0.5 < \beta < -0.25 \quad \beta \in (-0.5, -0.25)$

3. (i)

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single transformation represented by the matrix **A**.

(2)

The matrix **B** represents an enlargement, scale factor –2, with centre the origin.

(b) Write down the matrix **B**.

(1)

(ii)

$$\mathbf{M} = \begin{pmatrix} 3 & k \\ -2 & 3 \end{pmatrix}, \quad \text{where } k \text{ is a positive constant.}$$

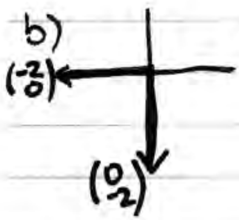
Triangle T has an area of 16 square units.

Triangle T is transformed onto the triangle T' by the transformation represented by the matrix **M**.

Given that the area of the triangle T' is 224 square units, find the value of k.

(3)

a)  rotation 45° anticlockwise centre (0,0)

b) 

B $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

ii) $\det M = 9 + 2k$

$T \to T' =$ enlargement sf $= 14$

$\therefore 9 + 2k = 14 \Rightarrow 2k = 5$

$\therefore \underline{\underline{k = 2.5}}$

4. The complex number z is given by

$$z = \frac{p + 2\mathrm{i}}{3 + p\mathrm{i}}$$

where p is an integer.

(a) Express z in the form $a + b\mathrm{i}$ where a and b are real. Give your answer in its simplest form in terms of p.

(4)

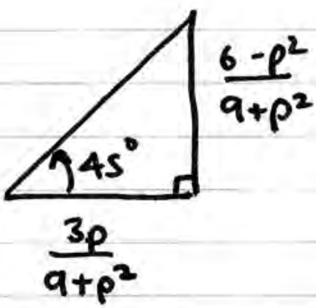
(b) Given that $\arg(z) = \theta$, where $\tan\theta = 1$ find the possible values of p.

(5)

a) $z = \frac{p+2i}{3+pi} \times \frac{(3-pi)}{(3-pi)} = \frac{5p+(6-p^2)i}{9+p^2}$

$z = \frac{5p}{9+p^2} + \frac{6-p^2}{9+p^2}i$

b) $\theta = \tan^{-1}(1) = \frac{\pi}{4}, \frac{5\pi}{4}$



$\therefore \frac{5p}{9+p^2} = \frac{6-p^2}{9+p^2}$

$\therefore 5p = 6 - p^2$

$\therefore p^2 + 5p - 6 = 0$

$\therefore (p+6)(p-1) = 0$

$p = -6 \quad p = 1$

5. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that

$$\sum_{r=1}^{n} r(r^2 - 3) = \frac{1}{4}n(n + 1)(n + 3)(n - 2)$$

(5)

(b) Calculate the value of $\sum_{r=10}^{50} r(r^2 - 3)$

(3)

$$\sum_{1}^{n} r(r^2-3) = \sum_{1}^{n} r^3 - 3\sum_{1}^{n} r = \frac{1}{4}n^2(n+1)^2 - 3\frac{1}{2}n(n+1)$$

$$= \frac{1}{4}n(n+1)[n(n+1) - 6]$$

$$= \frac{1}{4}n(n+1)(n^2+n-6)$$

$$= \frac{1}{4}n(n+1)(n+3)(n-2) \#$$

b) $\sum_{10}^{50} = \sum_{1}^{50} - \sum_{1}^{9} \Rightarrow \frac{1}{4}(50)(51)(53)(48) - \frac{1}{4}(9)(10)(12)(7)$

$= 1619910$

6.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Given that $\mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$,

(a) calculate the matrix $\mathbf{M}$,

(6)

(b) find the matrix $\mathbf{C}$ such that $\mathbf{MC} = \mathbf{A}$.

(4)

a) $M = (A+B)(2A-B) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -7 & -2 \end{pmatrix}$

b) $MC = A \Rightarrow M^{-1}MC = M^{-1}A \Rightarrow C = M^{-1}A$

$\det(M) = -2 - 7 = -9$

$\cancel{M^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix} \quad \therefore C = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}}$

$M^{-1} = -\frac{1}{9}\begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix}$

$C = \frac{-1}{9}\begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \frac{-1}{9}\begin{pmatrix} -5 & -2 \\ 13 & 7 \end{pmatrix}$

7. The parabola C has cartesian equation $y^2 = 4ax,\ a > 0$

The points $P(ap^2, 2ap)$ and $P'(ap^2, -2ap)$ lie on C.

(a) Show that an equation of the normal to C at the point P is

$$y + px = 2ap + ap^3$$

(5)

(b) Write down an equation of the normal to C at the point P'.

(1)

The normal to C at P meets the normal to C at P' at the point Q.

(c) Find, in terms of a and p, the coordinates of Q.

(2)

Given that S is the focus of the parabola,

(d) find the area of the quadrilateral $SPQP'$.

(3)

$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax) \Rightarrow 2y\frac{dy}{dx} = 4a \therefore \frac{dy}{dx} = \frac{4a}{2y}$

at $P(ap^2, 2ap)$ $\frac{dy}{dx} = \frac{4\cancel{a}}{4\cancel{a}p} = \frac{1}{p} \therefore m_n = -p$

$y - 2ap = -p(x - ap^2) \Rightarrow y - 2ap = -px + ap^3$

$\therefore y + px = 2ap + ap^3$ #

b)

at P', $m_n = -\left(\frac{-4ap}{4a}\right) = p$ $\quad y + 2ap = p(x - ap^2)$

$\therefore y - px = -2ap - ap^3$

c) Must meet at x-axis $\Rightarrow y = 0$ $\therefore px = 2ap + ap^3$

$x = 2a + ap^2$

$Q(2a + ap^2, 0)$

d) $S(a, 0)$

$\therefore$ Area $= \frac{1}{2}\vec{SQ} \times \vec{P'P}$

$= \frac{1}{2}(2a + ap^2 - a)(4ap) = 2ap(a + ap^2) = 2a^2p(1 + p^2)$

8. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant.

The point $P\left(ct, \frac{c}{t}\right)$, $t \neq 0$, is a general point on H.

An equation for the tangent to H at P is given by

$$y = -\frac{1}{t^2}x + \frac{2c}{t}$$

The points A and B lie on H.

The tangent to H at A and the tangent to H at B meet at the point $\left(-\frac{6}{7}c, \frac{12}{7}c\right)$.

Find, in terms of c, the coordinates of A and the coordinates of B.

(5)

$$\frac{12c}{7} = -\frac{-6c}{7t^2} + \frac{2c}{t} = \frac{6c}{7t^2} + \frac{14ct}{7t^2}$$

$$\therefore \frac{12ct^2}{7t^2} = \frac{6c + 14ct}{7t^2} \quad \therefore 12\cancel{c}t^2 = 6\cancel{c} + 14\cancel{c}t$$

$$\Rightarrow 6t^2 - 7t - 6 = 0$$

$$(3t+1)(2t-3) = 0$$

$$\therefore t = -\frac{1}{3} \quad t = \frac{3}{2}$$

$t = -\frac{1}{3}$ $A\left(-\frac{1}{3}c, -3c\right)$

$t = \frac{3}{2}$ $B\left(\frac{3}{2}c, \frac{2c}{3}\right)$

9. (a) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n2^n$$

(5)

(b) A sequence of numbers is defined by

$$u_1 = 0, \quad u_2 = 32,$$

$$u_{n+2} = 6u_{n+1} - 8u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 4^{n+1} - 2^{n+3}$$

(7)

a) $n=1 \quad \sum_{1}^{1}(r+1)2^{r-1} = (1+1)2^{1-1} = 2\times 2^0 = 2$

$n2^n = 1(2^1) = 2 \qquad \therefore$ true for $n=1$

assume true for $n=k \qquad \therefore \sum_{1}^{k}(r+1)2^{r-1} = k2^k$

$n=k+1 \quad \sum_{1}^{k+1}(r+1)2^{r-1}$

$= (k+1+1)2^{k+1-1} + \sum_{1}^{k}(r+1)2^{r-1}$

$= (k+2)2^k + k2^k = (2k+2)2^k = 2(k+1)2^k$

$= (k+1)\times 2 \times 2^k$

$= (k+1)2^{k+1}$

$n=k+1 \quad n2^n = (k+1)2^{k+1}$

$\therefore$ true for $n=1$, if true for $n=k$, true for $n=k+1$

$\therefore$ by Mathematical Induction true for $n \in \mathbb{Z}^+$

b) $n=1$ $u_1=0$ $u_1 = 4^{1+1} - 2^{1+3} = 4^2 - 2^4 = 16-16=0$

$\therefore$ true for $n=1$

$n=2$ $u_2 = 32$ $u_2 = 4^{2+1} - 2^{2+3} = 4^3 - 2^5 = 32$ $\therefore$ true for $n=2$

assume true for $n=k$ <u>and</u> $n=k+1$

$\therefore u_k = 4^{k+1} - 2^{k+3}$

$u_{k+1} = 4^{k+2} - 2^{k+4}$

$n=k+2$

$u_{k+2} = 4^{k+3} - 2^{k+5}$

$$\begin{aligned}
u_{k+2} &= 6u_{k+1} - 8u_k \\
&= 6(4^{k+2} - 2^{k+4}) - 8(4^{k+1} - 2^{k+3}) \\
&= 6\times 4^{k+2} - 6\times 2^{k+4} - 8\times 4^{k+1} + 8\times 2^{k+3} \\
&= 6\times 4^{k+2} - 3\times 2^{k+5} - 2\times 4^{k+2} + 2\times 2^{k+5} \\
&= 4\times 4^{k+2} - 1\times 2^{k+5} \\
&= 4^{k+3} - 2^{k+5} \quad \#
\end{aligned}$$

$\therefore$ true for $n=1, n=2$

If true for $n=k, n=k+1$ then true for $n=k+2$

$\therefore$ by Mathematical Induction true for $n \in \mathbb{Z}^+$