C4 S14 (R)

1. (a) Find the binomial expansion of

$$\frac{1}{\sqrt{(9-10x)}}, \qquad |x| < \frac{9}{10}$$

in ascending powers of x up to and including the term in x^2.
Give each coefficient as a simplified fraction.

(5)

(b) Hence, or otherwise, find the expansion of

$$\frac{3+x}{\sqrt{(9-10x)}}, \qquad |x| < \frac{9}{10}$$

in ascending powers of x, up to and including the term in x^2.
Give each coefficient as a simplified fraction.

(3)

a) $(9-10x)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}\left(1-\frac{10}{9}x\right)^{-\frac{1}{2}} = \frac{1}{3}\left(1-\frac{10}{9}x\right)^{-\frac{1}{2}}$

$= \frac{1}{3}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{10}{9}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{10}{9}x\right)^2\right]$

$= \frac{1}{3} + \frac{5}{27}x + \frac{25}{162}x^2$

b) $(3+x)\left[\frac{1}{3} + \frac{5}{27}x + \frac{25}{162}x^2\right)$

$\times 3$ $\quad 1 + \frac{5}{9}x + \frac{25}{54}x^2$

$\times x$ $\quad \frac{\cancel{3}\,1}{9\,\cancel{3}}x + \frac{10}{54}x^2$

$1 + \frac{8}{9}x + \frac{35}{54}x^2$

2.

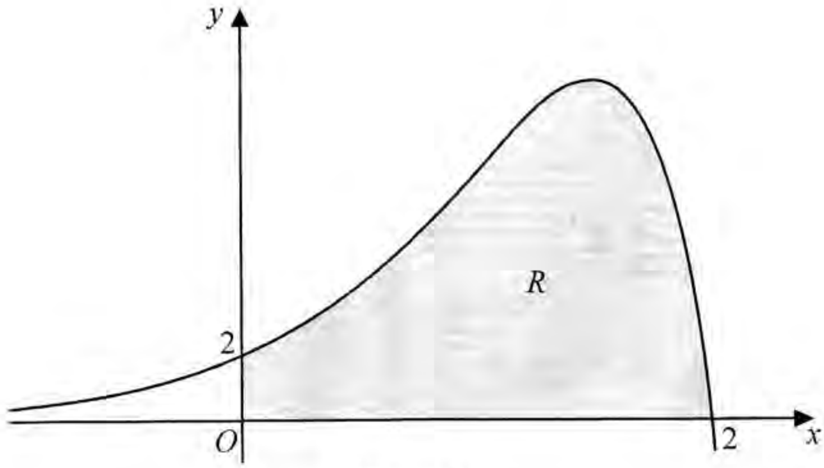

Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = (2 - x)e^{2x}, \qquad x \in \mathbb{R}$$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the y-axis.

The table below shows corresponding values of x and y for $y = (2 - x)e^{2x}$

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x	0	0.5	1	1.5	2
y	2	4.077	7.389	10.043	0

(a) Use the trapezium rule with all the values of y in the table, to obtain an approximation for the area of R, giving your answer to 2 decimal places.
(3)

(b) Explain how the trapezium rule can be used to give a more accurate approximation for the area of R.
(1)

(c) Use calculus, showing each step in your working, to obtain an exact value for the area of R. Give your answer in its simplest form.
(5)

a) $\frac{1}{2}(\frac{1}{2})(2+2(4.077 \ldots)+0) \triangleq \underline{5.88}$

b) More strips

c) $\int_0^2 (2-x)e^{2x}\,dx$ $\quad u = 2-x \quad v = \frac{1}{2}e^{2x}$ $\quad u' = -1 \quad v' = e^{2x}$ $\Rightarrow \frac{(2-x)}{2}e^{2x} + \frac{1}{2}\int e^{2x}\,dx$

$= \left[\left(\frac{2-x}{2}\right)e^{2x} + \frac{1}{4}e^{2x}\right]_0^2 = \left[\left(\frac{5}{4} - \frac{x}{2}\right)e^{2x}\right]_0^2 = \frac{1}{4}e^4 - \frac{5}{4}$

3.

$$x^2 + y^2 + 10x + 2y - 4xy = 10$$

(a) Find $\frac{dy}{dx}$ in terms of x and y, fully simplifying your answer. **(5)**

(b) Find the values of y for which $\frac{dy}{dx} = 0$ **(5)**

a) $\frac{d}{dx}(x^2+y^2+10x+2y-4xy) = \frac{d}{dx}(10)$

$= 2x + 2y\frac{dy}{dx} + 10 + 2\frac{dy}{dx} - 4x\frac{dy}{dx} - 4y = 0$

$2x - 4y + 10 = (4x - 2y - 2)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2x-4y+10}{4x-2y-2}$

b) $\frac{dy}{dx} = 0 \Rightarrow 2x - 4y + 10 = 0 \quad \therefore x = 2y - 5$

$\therefore y = \frac{1}{2}x + \frac{5}{2} \quad x^2 = 4y^2 - 20y + 25$

$(4y^2 - 20y + 25) + y^2 + 20y - 50 + 2y - 8y^2 + 20y = 10$

$-3y^2 + 22y = 35 \Rightarrow 3y^2 - 22y + 35 = 0$

$(3y - 7)(y - 5) = 0$

$\therefore y = \frac{7}{3} \quad y = \frac{5}{2}$

4. (a) Express $\frac{25}{x^2(2x+1)}$ in partial fractions.

(4)

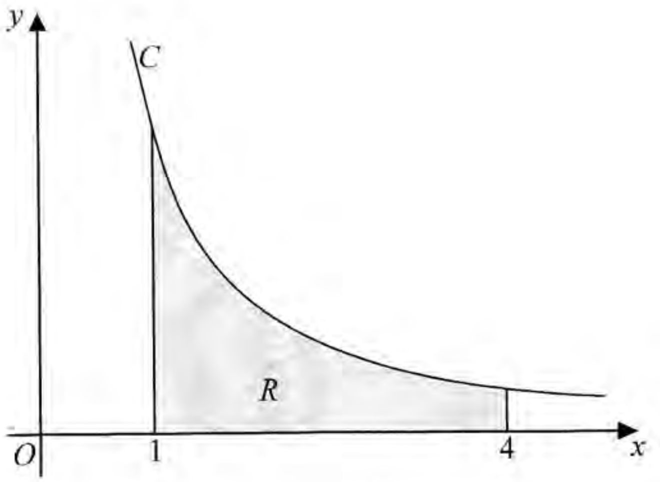


Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = \frac{5}{x\sqrt{(2x+1)}}$, $x > 0$

The finite region R is bounded by the curve C, the x-axis, the line with equation $x = 1$ and the line with equation $x = 4$

This region is shown shaded in Figure 2

The region R is rotated through 360° about the x-axis.

(b) Use calculus to find the exact volume of the solid of revolution generated, giving your answer in the form $a + b\ln c$, where a, b and c are constants.

(6)

a) $\frac{25}{x^2(2x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+1} \Rightarrow 25 = Ax(2x+1) + B(2x+1) + Cx^2$

$x = -\frac{1}{2} \Rightarrow 25 = \frac{1}{4}C \therefore C = 100$

$x = 0 \Rightarrow B = 25$

$x = 1 \quad 25 = 3A + 3B + C$

$25 = 3A + 75 + 100 \quad \therefore 3A = -150 \quad \therefore A = -50$

$= -\frac{50}{x} + \frac{25}{x^2} + \frac{100}{2x+1}$

b) $\text{Vol} = \pi\int y^2\,dx = \pi\int_1^4 -\frac{50}{x} + 25x^{-2} + 50\left(\frac{2}{2x+1}\right)dx$

$= \pi\left[-50\ln x - \frac{25}{x} + 50\ln(2x+1)\right]_1^4 = 50\left[\ln\left(\frac{2x+1}{x}\right) - \frac{1}{2x}\right]_1^4 \pi$

$= 50\pi\left[\left(\ln\frac{9}{4} - \frac{1}{8}\right) - \left(\ln 3 - \frac{1}{2}\right)\right] = 50\pi\left(\ln\frac{3}{4} + \frac{3}{8}\right) = \frac{75\pi}{4} + 50\pi\ln\left(\frac{3}{4}\right)$

5. At time t seconds the radius of a sphere is r cm, its volume is V cm^3 and its surface area is S cm^2.

$\left[\textit{You are given that } V = \frac{4}{3}\pi r^3 \textit{ and that } S = 4\pi r^2\right]$

The volume of the sphere is increasing uniformly at a constant rate of 3 cm^3 s^{-1}.

(a) Find $\frac{\mathrm{d}r}{\mathrm{d}t}$ when the radius of the sphere is 4 cm, giving your answer to 3 significant figures.

(4)

(b) Find the rate at which the surface area of the sphere is increasing when the radius is 4 cm.

(2)

a) $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ $\qquad \frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = 3$

$\therefore \frac{dr}{dt} = \left(\frac{1}{4\pi r^2}\right) \times 3$

$r = 4 \quad \frac{dr}{dt} = 0.0149$

2

b) find $\frac{ds}{dt} = \frac{ds}{dr} \times \frac{dr}{dt}$ $\qquad S = 4\pi r^2$

$\frac{ds}{dr} = 8\pi r$

$\frac{ds}{dt} = (\cancel{8}^{2}\cancel{\pi}\cancel{r})\left(\frac{3}{\cancel{4}\cancel{\pi}r^{\cancel{2}}}\right) = \frac{6}{r}$

when $r = 4 \quad \frac{ds}{dt} = 1.5$

2

6. With respect to a fixed origin, the point A with position vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ lies on the line l_1 with equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \text{where } \lambda \text{ is a scalar parameter,}$$

and the point B with position vector $4\mathbf{i} + p\mathbf{j} + 3\mathbf{k}$, where p is a constant, lies on the line l_2 with equation

$$\mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}, \quad \text{where } \mu \text{ is a scalar parameter.}$$

(a) Find the value of the constant p.

(1)

(b) Show that l_1 and l_2 intersect and find the position vector of their point of intersection, C.

(4)

(c) Find the size of the angle ACB, giving your answer in degrees to 3 significant figures.

(3)

(d) Find the area of the triangle ABC, giving your answer to 3 significant figures.

(2)

a) $a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $l_1 = \begin{pmatrix} 1 \\ 2+2\lambda \\ 3-\lambda \end{pmatrix}$ $b = \begin{pmatrix} 4 \\ p \\ 3 \end{pmatrix}$ $l_2 = \begin{pmatrix} 7+3\mu \\ -5\mu \\ 7+4\mu \end{pmatrix}$

$\begin{pmatrix} 7+3\mu \\ -5\mu \\ 7+4\mu \end{pmatrix} = \begin{pmatrix} 4 \\ p \\ 3 \end{pmatrix}$ $7+3\mu = 4 \Rightarrow 3\mu = -3 \therefore \mu = -1$

$-5\mu = p \quad \therefore \underline{p = 5}$

b) $\begin{pmatrix} 1 \\ 2+2\lambda \\ 3-\lambda \end{pmatrix} = \begin{pmatrix} 7+3\mu \\ -5\mu \\ 7+4\mu \end{pmatrix}$ $1 = 7+3\mu \Rightarrow 3\mu = -6 \therefore \mu = -2$

$2+2\lambda = 10 \Rightarrow 2\lambda = 8 \therefore \lambda = 4$

check $3-\lambda = -1 \quad 7+4\mu = -1$

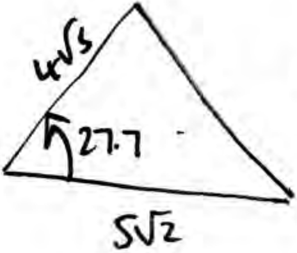
$\lambda = 4$ $\begin{pmatrix} 1 \\ 10 \\ -1 \end{pmatrix} = C$ $\therefore$ they intersect.

c) $\vec{CA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -8 \\ 4 \end{pmatrix}$ $\vec{CB} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

$$|\vec{CA}| = \sqrt{8^2+4^2} = 4\sqrt{5}$$
$$|\vec{CB}| = \sqrt{3^2+5^2+4^2} = 5\sqrt{2}$$
$$\vec{CA}\cdot\vec{CB} = 0+40+16 = 56$$

$$\theta = \cos^{-1}\left(\frac{56}{4\sqrt{5}\times 5\sqrt{2}}\right)$$

$$\theta = 27.69456\ldots$$
$$\theta = 27.7.$$



$$\text{Area} = \tfrac{1}{2}ab\sin C$$
$$\tfrac{1}{2}(5\sqrt{2})(4\sqrt{5})\sin 27.7$$

$$\text{Area} = 14.7$$

7. The rate of increase of the number, N, of fish in a lake is modelled by the differential equation

$$\frac{dN}{dt} = \frac{(kt-1)(5000-N)}{t} \qquad t > 0, \quad 0 < N < 5000$$

In the given equation, the time t is measured in years from the start of January 2000 and k is a positive constant.

(a) By solving the differential equation, show that

$$N = 5000 - Ate^{-kt}$$

where A is a positive constant.

(5)

After one year, at the start of January 2001, there are 1200 fish in the lake.

After two years, at the start of January 2002, there are 1800 fish in the lake.

(b) Find the exact value of the constant A and the exact value of the constant k.

(4)

(c) Hence find the number of fish in the lake after five years. Give your answer to the nearest hundred fish.

(1)

a) $-\int \frac{-1}{5000-N}\,dN = \int \frac{kt-1}{t}\,dt = \int k - \frac{1}{t}\,dt$

$-\ln(5000-N) = kt - \ln t + c \quad \therefore \ln(5000-N) = \ln t - kt + b$

$\Rightarrow 5000 - N = e^{\ln t - kt + b} = \frac{e^{\ln t} \times e^{b}}{e^{kt}} = Ate^{-kt}$

$\therefore N = 5000 - Ate^{-kt}$

b) $t=1 \quad N = 5000 - Ae^{-k} = 1200 \quad \therefore Ae^{-k} = 3800$

$t=2 \quad N = 5000 - 2Ae^{-2k} = 1800 \quad \therefore Ae^{-2k} = 1600$

$\Rightarrow e^{k} = \frac{19}{8}$

$\therefore k = \ln\left(\frac{19}{8}\right)$

$A = 3800e^{k} = 3800e^{\ln\left(\frac{19}{8}\right)}$

$\therefore A = 9025$

c) $N = 5000 - 9025te^{-\ln\left(\frac{19}{8}\right)t}$

$t = 5 \Rightarrow N = 4402.828$

$\therefore N \approx 4400$

8.

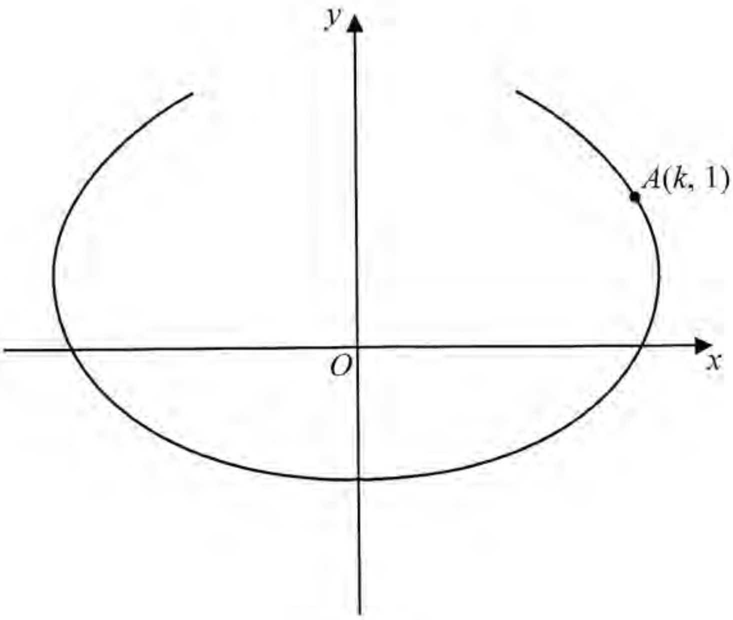


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = t - 4\sin t,\ y = 1 - 2\cos t, \qquad -\frac{2\pi}{3} \leqslant t \leqslant \frac{2\pi}{3}$$

The point A, with coordinates $(k, 1)$, lies on the curve.

Given that $k > 0$

(a) find the exact value of k,

(2)

(b) find the gradient of the curve at the point A.

(4)

There is one point on the curve where the gradient is equal to $-\frac{1}{2}$

(c) Find the value of t at this point, showing each step in your working and giving your answer to 4 decimal places.

[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(6)

a) $y=1 \Rightarrow 2\cos t = 0 \Rightarrow t = \pm\frac{\pi}{2} \quad \therefore x = -\frac{\pi}{2} - 4\sin\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 4$

Since $k>0$

b) $\frac{dy}{dt} = 2\sin t \qquad \therefore \frac{dy}{dx} = \frac{2\sin t}{1-4\cos t} = \frac{-2}{1} = -2$

$\frac{dx}{dt} = 1 - 4\cos t$

$t = -\frac{\pi}{2}$

c) $\dfrac{2\sin t}{1-4\cos t} = -\dfrac{1}{2} \Rightarrow 4\sin t = 4\cos t - 1$

$$16\sin^2 t = 16\cos^2 t - 8\cos t + 1$$

$$16 - 16\cos^2 t = 16\cos^2 t - 8\cos t + 1$$

$$\Rightarrow 32\cos^2 t - 8\cos t - 15 = 0$$

$$\therefore \cos t = \frac{8 \pm \sqrt{64 - 4(32)(-15)}}{64}$$

$$\Rightarrow \cos t = 0.82097\ldots \qquad \cos t = -0.57097$$

$$\frac{1+\sqrt{31}}{8} \qquad \frac{1-\sqrt{31}}{8}$$

$$t = 0.6077^c \; ; \; 2.1785^c$$

$$-\frac{2\pi}{3} \le t \le \frac{2\pi}{3}$$

$$-2.094 \le t \le 2.094$$

$$\therefore t = 0.6077^c$$

2