S14 C3(R)

1. Express

$$\frac{3}{2x+3} - \frac{1}{2x-3} + \frac{6}{4x^2-9}$$

as a single fraction in its simplest form. $= (2x-3)(2x+3)$

(4)

$$\frac{3(2x-3)-(2x+3)+6}{(2x+3)(2x-3)} = \frac{4x-6}{(2x+3)(2x-3)} = \frac{2(x-\cancel{3})}{(2x+3)(2x-\cancel{3})}$$

$$= \frac{2}{2x+3} \quad \#$$

2. A curve C has equation $y = e^{4x} + x^4 + 8x + 5$

(a) Show that the x coordinate of any turning point of C satisfies the equation

$$x^3 = -2 - e^{4x}$$

(3)

(b) On the axes given on page 5, sketch, on a single diagram, the curves with equations

(i) $y = x^3$,

(ii) $y = -2 - e^{4x}$

On your diagram give the coordinates of the points where each curve crosses the y-axis and state the equation of any asymptotes.

(4)

(c) Explain how your diagram illustrates that the equation $x^3 = -2 - e^{4x}$ has only one root.

(1)

The iteration formula

$$x_{n+1} = (-2 - e^{4x_n})^{\frac{1}{3}}, \qquad x_0 = -1$$

can be used to find an approximate value for this root.

(d) Calculate the values of x_1 and x_2, giving your answers to 5 decimal places.

(2)

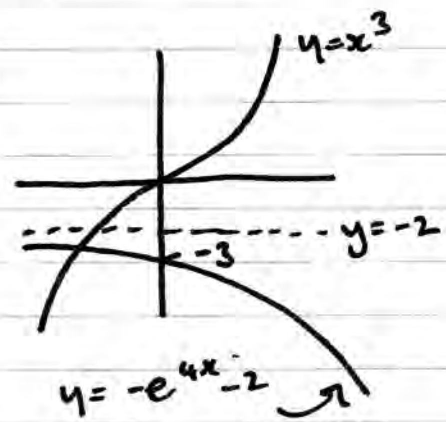
(e) Hence deduce the coordinates, to 2 decimal places, of the turning point of the curve C.

(2)

a) $\frac{dy}{dx} = 4e^{4x} + 4x^3 + 8$ TP $\frac{dy}{dx} = 0 \Rightarrow 4x^3 = -8 - 4e^{4x}$

$\therefore x^3 = -2 - e^{4x}$ #

b)



c) Only one point of intersection.

d) $x_0 = -1$

$x_1 = -1.26376$

$x_2 = -1.26126$

e $(-1.26, -2.55)$

Leave blank

3. (i) (a) Show that $2\tan x - \cot x = 5\operatorname{cosec} x$ may be written in the form

$$a\cos^2 x + b\cos x + c = 0$$

stating the values of the constants a, b and c.

(4)

(b) Hence solve, for $0 \leqslant x < 2\pi$, the equation

$$2\tan x - \cot x = 5\operatorname{cosec} x$$

giving your answers to 3 significant figures.

(4)

(ii) Show that

$$\tan\theta + \cot\theta \equiv \lambda\operatorname{cosec} 2\theta, \qquad \theta \neq \frac{n\pi}{2}, \quad n \in \mathbb{Z}$$

stating the value of the constant λ.

(4)

a) $\dfrac{2\sin x}{\cos x}{}^{\times \sin x}_{\times \sin x} - \dfrac{\cos x}{\sin x}{}^{\times \cos x}_{\times \cos x} = \dfrac{5}{\sin x}{}^{\times\cos x}_{\times \cos x} = \dfrac{2\sin^2 x - \cos^2 x}{\sin x\cos x} = \dfrac{5\cos x}{\sin x \cos x}$

$\therefore\ 2(1-\cos^2 x) - \cos^2 x = 5\cos x \Rightarrow 3\cos^2 x + 5\cos x - 2 = 0$

$a = 3 \quad b = 5 \quad c = -2$

b) $(3\cos x - 1)(\cos x + 2) = 0 \Rightarrow \cos x = \frac{1}{3} \quad x = \cos^{-1}(\frac{1}{3})$

No solution

$x = 1.23^c,\ 5.05^c$

ii) $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta\cos\theta}$

$= \dfrac{2}{2\sin\theta\cos\theta} = \dfrac{2}{\sin 2\theta} = 2\operatorname{cosec}^2 2\theta \qquad \lambda = 2$

4. (i) Given that

$$x = \sec^2 2y, \qquad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{dy}{dx} = \frac{1}{4x\sqrt{(x-1)}}$$

(4)

(ii) Given that

$$y = (x^2 + x^3)\ln 2x$$

find the exact value of $\frac{dy}{dx}$ at $x = \frac{e}{2}$, giving your answer in its simplest form.

(5)

(iii) Given that

$$f(x) = \frac{3\cos x}{(x+1)^{\frac{1}{3}}}, \qquad x \neq -1$$

show that

$$f'(x) = \frac{g(x)}{(x+1)^{\frac{4}{3}}}, \qquad x \neq -1$$

where g(x) is an expression to be found.

(3)

i) $x = (\sec 2y)^2$ $\quad \frac{dx}{dy} = 2(\sec 2y) \times 2\sec 2y \tan 2y$

$= 4\sec^2 2y \tan 2y$

$\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$ $\quad \therefore \frac{dy}{dx} = \frac{1}{4\sec^2 2y \tan 2y}$

$\tan^2 y + 1 = \sec^2 y$

$\tan y = \sqrt{\sec^2 y - 1}$ $\quad \therefore \frac{dy}{dx} = \frac{1}{4x\sqrt{x-1}}$

ii) $u = x^2 + x^3$ $\quad v = \ln 2x$ $\quad \frac{dy}{dx} = (2x + 3x^2)\ln 2x + \frac{x^2 + x^3}{x}$

$u' = 2x + 3x^2$ $\quad v' = \frac{2}{2x} = \frac{1}{x}$ $\quad = x(2+3x)\ln 2x + x + x^2$

$x = \frac{e}{2}$ $\quad \ln 2x = \ln e = 1$ $\quad \Rightarrow \frac{dy}{dx}\bigg|_{\frac{e}{2}} = \frac{e}{2}\left(2 + \frac{3e}{2}\right) + \frac{e}{2} + \frac{e^2}{4}$

$= \frac{3}{2}e + \frac{4}{4}e^2 = \frac{3}{2}e\left(1 + \frac{2}{3}e\right)$

(iii) $u = 3\cos x \quad v = (x+1)^{\frac{1}{3}}$

$u' = -3\sin x \quad v' = \frac{1}{3}(x+1)^{-\frac{2}{3}}$

$$\frac{-3(x+1)^{\frac{1}{3}}\sin x - (x+1)^{-\frac{2}{3}}\cos x}{(x+1)^{\frac{2}{3}}}$$

$$= \frac{(x+1)^{-\frac{2}{3}}\left[-3(x+1)\sin x - \cos x\right]}{(x+1)^{\frac{2}{3}}}$$

$$= \frac{-3(x+1)\sin x - \cos x}{(x+1)^{\frac{4}{3}}}$$

5. (a) Sketch the graph with equation

$$y = |4x - 3|$$

stating the coordinates of any points where the graph cuts or meets the axes.

(2)

Find the complete set of values of x for which

(b)

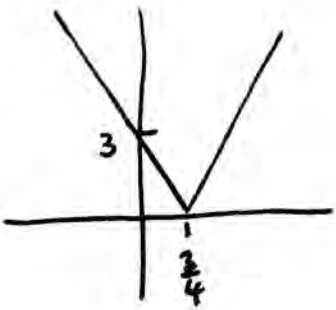
$$|4x - 3| > 2 - 2x$$

(4)

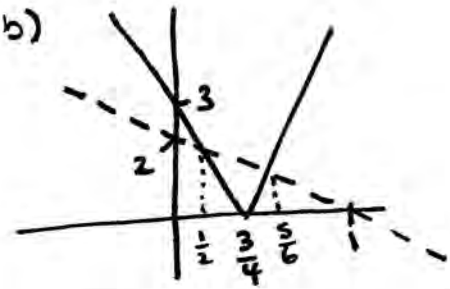
(c)

$$|4x - 3| > \frac{3}{2} - 2x$$

(2)



b)



$4x - 3 = 2 - 2x$

$6x = 5$

$x = \frac{5}{6}$

$4x - 3 = 2x - 2$

$2x = 1$

$x = \frac{1}{2}$

$x < \frac{1}{2}$ or $x > \frac{5}{6}$

c) $4x - 3 = \frac{3}{2} - 2x$

$6x = \frac{9}{2}$

$x = \frac{9}{12} = \frac{3}{4}$

$x \in \mathbb{R}$

$x \neq \frac{3}{4}$

6. The function f is defined by

$$f : x \to e^{2x} + k^2, \quad x \in \mathbb{R}, \quad k \text{ is a positive constant.}$$

(a) State the range of f.

(1)

(b) Find f^{-1} and state its domain.

(3)

The function g is defined by

$$g : x \to \ln(2x), \qquad x > 0$$

(c) Solve the equation

$$g(x) + g(x^2) + g(x^3) = 6$$

giving your answer in its simplest form.

(4)

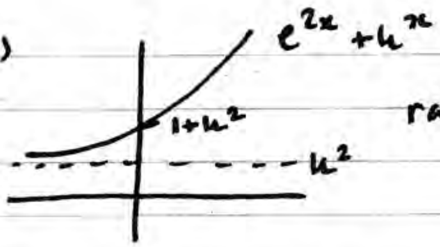
(d) Find fg(x), giving your answer in its simplest form.

(2)

(e) Find, in terms of the constant k, the solution of the equation

$$fg(x) = 2k^2$$

(2)

a)


range $y > k^2$

b) $x = e^{2y} + k^2 \Rightarrow e^{2y} = x - k^2 \Rightarrow 2y = \ln|x - k^2|$

$\therefore y = \frac{1}{2}\ln|x - k^2| = f^{-1}(x)$

$x > k^2$
domain

c) $g(x) + g(x^2) + g(x^3) = \ln(2x) + \ln(2x^2) + \ln(2x^3)$

$= \ln(2x \times 2x^2 \times 2x^3) = \ln(8x^6) = 6$

$8x^6 = e^6 \quad \therefore x = \sqrt[6]{\frac{1}{8}e^6} = \frac{1}{\sqrt{2}}e$

d) $fg(x) = f(\ln(2x)) = e^{2\ln(2x)} + k^2 = (2x)^2 + k^2 = 4x^2 + k^2$

e) $4x^2 + k^2 = 2k^2 \Rightarrow 4x^2 = k^2 \Rightarrow x^2 = \frac{1}{4}k^2 \quad \therefore x = \pm\frac{1}{2}k \quad \therefore x = \frac{1}{2}k$

7.

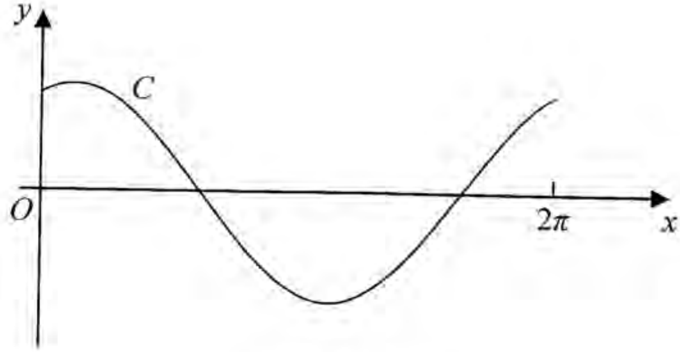


Figure 1

Figure 1 shows the curve C, with equation $y = 6\cos x + 2.5\sin x$ for $0 \leqslant x \leqslant 2\pi$

(a) Express $6\cos x + 2.5\sin x$ in the form $R\cos(x - \alpha)$, where R and α are constants with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give your value of α to 3 decimal places.

(3)

(b) Find the coordinates of the points on the graph where the curve C crosses the coordinate axes.

(3)

A student records the number of hours of daylight each Sunday throughout the year. She starts on the last Sunday in May with a recording of 18 hours, and continues until her final recording 52 weeks later.

She models her results with the continuous function given by

$$H = 12 + 6\cos\left(\frac{2\pi t}{52}\right) + 2.5\sin\left(\frac{2\pi t}{52}\right), \quad 0 \leqslant t \leqslant 52$$

where H is the number of hours of daylight and t is the number of weeks since her first recording.

Use this function to find

(c) the maximum and minimum values of H predicted by the model,

(3)

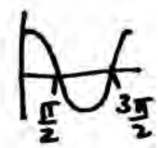
(d) the values for t when $H = 16$, giving your answers to the nearest whole number.

[*You must show your working. Answers based entirely on graphical or numerical methods are not acceptable.*]

(6)

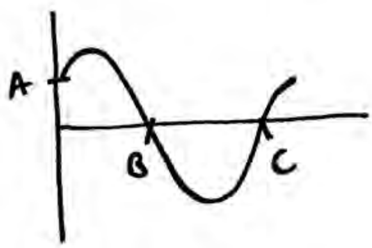
7a) $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$

$\qquad\qquad\qquad\quad 6\cos x \quad + 2.5\sin x$



$\dfrac{R\sin\alpha = 2.5}{R\cos\alpha = 6} \Rightarrow \tan\alpha = \frac{5}{12} \quad \alpha = 0.395$

$R^2 = 2.5^2 + 6^2 \quad \therefore R = 6.5 \qquad 6.5\cos(x - 0.395\ldots)$



$x = 0 \quad y = 6 \quad A(0,6) \qquad B, C \xrightarrow{} 0.395\ldots$

$B(1.97, 0)$

$C(5.11, 0)$

c) $H = 12 + 6.5\cos(x - 0.395\ldots)$ when $x = \frac{2\pi t}{52}$

$6.5\cos(x - 0.395)$ $\nearrow$ max $= 6.5$ when $x = 0.395, 2\pi + 0.395\ldots$

$\qquad\qquad\qquad\quad$ $\searrow$ min $= -6.5$ when $x = \frac{\pi}{2} + 0.395, \frac{3\pi}{2} + 0.395\ldots$

$H_{max} = 18.5 \qquad H_{min} = 5.5$

d) $16 = 12 + 6.5\cos(x - 0.395) \Rightarrow \cos(x - 0.395) = \frac{4}{6.5}$

$\therefore x - 0.395 = 0.9079, 5.375 \to +2\pi$ etc.

$\therefore x = 1.3027, 5.77, \ldots$

$\frac{2\pi t}{52} = \nearrow$

$(\times 52) \div (2\pi) \quad t = 10.78 \ ; 47.75$

$t = 11, 48$