

1. Find the value of

SI4 F11AL

$$\sum_{r=1}^{200} (r+1)(r-1)$$

(4)

$$\begin{aligned} \sum_1^n r^2 - 1 &= \sum_1^n r^2 - \sum_1^n 1 = \frac{1}{6}n(n+1)(2n+1) - n = \frac{1}{6}n[2n^2+3n+1-6] \\ &= \frac{1}{6}n[2n^2+3n-5] = \frac{1}{6}n(n-1)(2n+5) \end{aligned}$$

$$\therefore \sum_1^{200} r^2 - 1 = \frac{1}{6}(200)(199)(405) = 2686500$$

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2. Given that $-2 + 3i$ is a root of the equation

$$z^2 + pz + q = 0$$

where p and q are real constants,

(a) write down the other root of the equation.

(1)

(b) Find the value of p and the value of q .

(3)

$$z = -2 + 3i \Rightarrow z^* = -2 - 3i$$

$$(z + 2 - 3i)(z + 2 + 3i) = 0$$

$$z^2 + 4z + (4+9) = 0 \Rightarrow z^2 + 4z + 13 = 0$$

3.

$$A = \begin{pmatrix} 4 & -2 \\ a & -3 \end{pmatrix}$$

where a is a real constant and $a \neq 6$

(a) Find A^{-1} in terms of a .

(3)

Given that $A + 2A^{-1} = I$, where I is the 2×2 identity matrix,

(b) find the value of a .

(3)

$$a) \quad A^{-1} = \frac{1}{\det A} \begin{pmatrix} -3 & 2 \\ -a & 4 \end{pmatrix} \quad \det A = -12 + 2a = 2(a-6)$$

$$\therefore A^{-1} = \frac{1}{2(a-6)} \begin{pmatrix} -3 & 2 \\ -a & 4 \end{pmatrix}$$

$$b) \quad \begin{pmatrix} 4 & -2 \\ a & -3 \end{pmatrix} + \frac{1}{a-6} \begin{pmatrix} -3 & 2 \\ -a & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$-2 + \frac{2}{a-6} = 0 \quad \therefore \frac{2}{a-6} = 2 \quad \therefore a-6=1$$

$$\therefore a=7$$

$$f(x) = x^{\frac{3}{2}} - 3x^{\frac{1}{2}} - 3, \quad x > 0$$

Given that α is the only real root of the equation $f(x) = 0$,

(a) show that $4 < \alpha < 5$

(2)

(b) Taking 4.5 as a first approximation to α , apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α , giving your answer to 3 decimal places.

(5)

(c) Use linear interpolation once on the interval $[4, 5]$ to find another approximation to α , giving your answer to 3 decimal places.

(3)

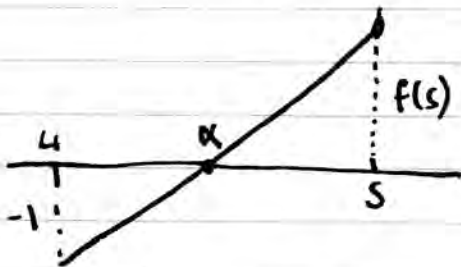
$$\begin{aligned} \text{a) } f(4) &= 4^{\frac{3}{2}} - 3 \cdot 4^{\frac{1}{2}} - 3 = (8) - 3(2) - 3 = -1 < 0 \\ f(5) &= \sqrt{125} - 3\sqrt{5} - 3 = 2\sqrt{5} - 3 = 1.47 > 0 \end{aligned}$$

\therefore by sign change rule $\alpha \in (4, 5)$

$$\text{b) } \alpha = 4.5 - \frac{f(4.5)}{f'(4.5)} = 4.426$$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}$$

c)



$$\frac{5-\alpha}{f(5)} = \frac{\alpha-4}{1}$$

$$5-\alpha = 1.4721 \dots \alpha - 5.888 \dots$$

$$10.8885 \dots = 2.4721 \dots \alpha$$

$$\alpha = \frac{4.405}{2}$$

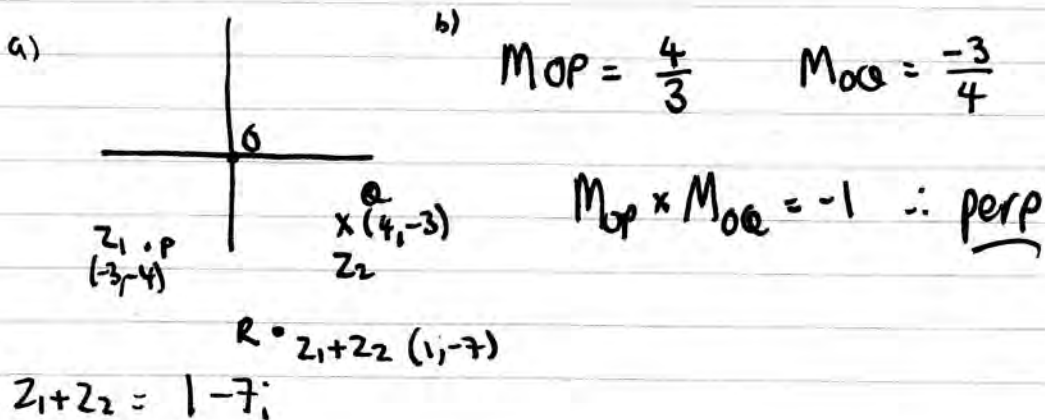
5. Given that $z_1 = -3 - 4i$ and $z_2 = 4 - 3i$

(a) show, on an Argand diagram, the point P representing z_1 and the point Q representing z_2 (2)

(b) Given that O is the origin, show that OP is perpendicular to OQ . (2)

(c) Show the point R on your diagram, where R represents $z_1 + z_2$ (1)

(d) Prove that $OPRQ$ is a square. (2)



d) $m_{PR} = \frac{-3}{4} \quad \therefore \text{perp to } OP \text{ and Parallel to } OQ$

$$|OP| = |PR| = |RQ| = |OQ| = \sqrt{3^2 + 4^2} = 5$$

\therefore 4 sides equal in length

\therefore 4 interior angles must be 90°

\therefore Square.

6. It is given that α and β are roots of the equation $3x^2 + 5x - 1 = 0$

(a) Find the exact value of $\alpha^3 + \beta^3$

(3)

(b) Find a quadratic equation which has roots $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$, giving your answer in the form $ax^2 + bx + c = 0$, where a , b and c are integers.

(5)

$$x^2 + \frac{5}{3}x - \frac{1}{3} = 0 \Rightarrow \left(x + \frac{5}{6}\right)^2 - \frac{25}{36} - \frac{12}{36} = 0$$

$$\Rightarrow \left(x + \frac{5}{6}\right) = \pm \frac{\sqrt{37}}{6} \quad x = -\frac{5}{6} \pm \frac{\sqrt{37}}{6}$$

$$6\alpha = -5 + \sqrt{37} = \sqrt{37} - 5 \quad 6\beta = \sqrt{37} + 5$$

$$216\alpha^3 = (\sqrt{37})^3 - 3(37)(5) + 3\sqrt{37}(25) - 125$$

$$216\beta^3 = (\sqrt{37})^3 + 3(37)(5) + 3\sqrt{37}(25) + 125$$

$$216(\alpha^3 + \beta^3) = 2(\sqrt{37})^3 + 150\sqrt{37}$$

$$\therefore \alpha^3 + \beta^3 = \frac{2\sqrt{37} [37 + 75]}{216} = \frac{28\sqrt{37}}{27}$$

$$x^2 + \frac{5}{3}x - \frac{1}{3} = 0 \quad x^2 + (\alpha + \beta)x + (\alpha\beta) = 0$$

$$\alpha + \beta = \frac{5}{3} \Rightarrow 3\alpha + 3\beta = 5 \quad \alpha\beta = -\frac{1}{3}$$

$$(\alpha + \beta)^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$$

$$\left(x - \frac{\alpha^2}{\beta}\right) \left(x - \frac{\beta^2}{\alpha}\right)$$

6. It is given that α and β are roots of the equation $3x^2 + 5x - 1 = 0$

(a) Find the exact value of $\alpha^3 + \beta^3$ (3)

(b) Find a quadratic equation which has roots $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers. (5)

$$3x^2 + 5x - 1 = 0 \Rightarrow x^2 + \frac{5}{3}x - \frac{1}{3} = 0$$

$$x^2 - (\alpha + \beta)x - (\alpha\beta) = 0$$

$$\begin{aligned} \therefore (\alpha + \beta) &= -\frac{5}{3} & (\alpha + \beta)^3 &= (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) \\ (\alpha + \beta)^2 &= \frac{25}{9} & &= (\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta) \\ \alpha\beta &= -\frac{1}{3} & &= \left(-\frac{5}{3}\right) \left(\frac{25}{9} - 3\left(-\frac{1}{3}\right)\right) \\ & & &= -\frac{5}{3} \left(\frac{34}{9}\right) = \frac{-170}{27} \end{aligned}$$

alt

$$\left(x + \frac{5}{6}\right)^2 - \frac{25}{36} - \frac{12}{36} \Rightarrow x = \frac{-5 \pm \sqrt{37}}{6}$$

$$6\alpha = -5 + \sqrt{37} \quad 6\beta = -5 - \sqrt{37}$$

$$\begin{aligned} \Rightarrow 216\alpha^3 &= (-5 + \sqrt{37})^3 = (\sqrt{37} - 5)^3 = 37\sqrt{37} - 3(37)(5) + 3(\sqrt{37})(25) - 125 \\ 216\beta^3 &= (-5 - \sqrt{37})^3 = (-\sqrt{37} - 5)^3 = -37\sqrt{37} - 3(37)(5) - 3(\sqrt{37})(25) - 125 \end{aligned}$$

$$\therefore 216(\alpha^3 + \beta^3) = -1110 - 250 \quad \therefore \alpha^3 + \beta^3 = \frac{-170}{27}$$

b) $x^2 - \left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right)x + \frac{(\alpha\beta)^2}{\alpha\beta} = 0$

$$x^2 - \left(\frac{\alpha^3 + \beta^3}{\alpha\beta}\right)x + (\alpha\beta) = 0$$

$$x^2 - \left(\frac{-170}{27} \cdot -\frac{1}{3}\right)x - \frac{1}{3} = 0 \Rightarrow x^2 - \frac{170}{9}x - \frac{1}{3} = 0$$

(x9)

$$\therefore 9x^2 - 170x - 3 = 0$$

7.

$$P = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

- (a) Describe fully the single geometrical transformation U represented by the matrix P . (3)

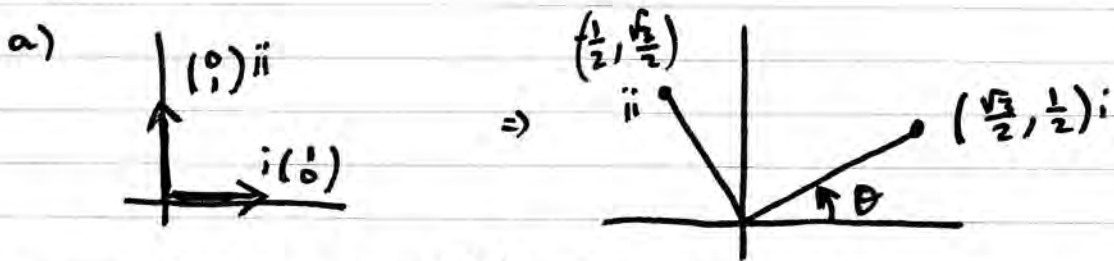
The transformation V , represented by the 2×2 matrix Q , is a reflection in the x -axis.

- (b) Write down the matrix Q . (1)

Given that V followed by U is the transformation T , which is represented by the matrix R ,

- (c) find the matrix R . (2)

- (d) Show that there is a real number k for which the transformation T maps the point $(1, k)$ onto itself. Give the exact value of k in its simplest form. (5)



$$\theta = \tan^{-1} \left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) = \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) = \frac{\pi}{6} \quad \therefore \text{anticlockwise rotation about } O, \text{ by } \frac{\pi}{6}.$$

b)

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

c)

$$R = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

d)

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} + \frac{1}{2}k \\ \frac{1}{2} - \frac{\sqrt{3}}{2}k \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix} \Rightarrow \frac{1}{2}k = 1 - \frac{\sqrt{3}}{2} \therefore k = 2 - \sqrt{3}$$

8. The hyperbola H has cartesian equation $xy = 16$
 The parabola P has parametric equations $x = 8t^2$, $y = 16t$.

(a) Find, using algebra, the coordinates of the point A where H meets P .

(3)

Another point $B(8, 2)$ lies on the hyperbola H .

(b) Find the equation of the normal to H at the point $(8, 2)$, giving your answer in the form $y = mx + c$, where m and c are constants.

(5)

(c) Find the coordinates of the points where this normal at B meets the parabola P .

(6)

$$\begin{aligned} \text{a) } xy &= 16 & xy &= 16(8t^3) & \therefore & 16 = 16(8t^3) \\ \therefore 8t^3 &= 1 & \therefore t^3 &= \frac{1}{8} & \therefore t &= \frac{1}{2} & \Rightarrow x = \frac{2}{2} \quad y = \frac{8}{2} \end{aligned}$$

$$\text{b) } y = \frac{16}{x} = 16x^{-1} \quad \frac{dy}{dx} = -16x^{-2} = -\frac{16}{x^2}$$

$$\text{at } (8, 2) \quad m_t = \frac{-16}{64} = -\frac{1}{4} \quad \therefore m_n = 4$$

$$y - 2 = 4(x - 8) \Rightarrow y - 2 = 4x - 32 \Rightarrow y = \frac{4x - 30}{2}$$

$$\text{c) } y = 4x - 30 \Rightarrow 16t = 32t^2 - 30$$

$$\Rightarrow 32t^2 - 16t - 30 = 0 \Rightarrow 16t^2 - 8t - 15 = 0$$

$$(4t - 5)(4t + 3) = 0$$

$$t = \frac{5}{4} \quad t = -\frac{3}{4}$$

$$t = \frac{5}{4} \quad x = 8t^2 = \frac{25}{2} \quad y = 16t = 20 \quad \left(\frac{25}{2}, 20\right)$$

$$t = -\frac{3}{4} \quad x = \frac{9}{2} \quad y = -12 \quad \left(\frac{9}{2}, -12\right)$$

9. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^n r(r+1)(r+2) = \frac{n(n+1)(n+2)(n+3)}{4} \quad (5)$$

(ii) Prove by induction that,

$$4^n + 6n + 8 \text{ is divisible by } 18$$

for all positive integers n .

(6)

$$n=1 \quad \text{LHS} = 1(2)(3) = 6 \quad \text{RHS} = \frac{1(2)(3)(4)}{4} = 6$$

\therefore true for $n=1$

$$\text{assume true for } n=k \quad \sum_{r=1}^k r(r+1)(r+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

$$\begin{aligned} n=k+1 \quad \sum_{r=1}^{k+1} r(r+1)(r+2) &= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\ &= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4} \\ &= \frac{(k+1)(k+2)(k+3)[k+4]}{4} \end{aligned}$$

$$n=k+1 \Rightarrow \frac{n(n+1)(n+2)(n+3)}{4} = \frac{(k+1)(k+2)(k+3)(k+4)}{4} \quad \therefore \text{LHS} = \text{RHS}$$

\therefore true for $n=1$, true for $n=k+1$ if true for $n=k$ \therefore by Mathematical Induction true for $n \in \mathbb{Z}^+$

$$\text{ii) } 4^n + 6n + 8$$

$$n=1 \Rightarrow 4^1 + 6(1) + 8 = 18 = 18 \times 1 \therefore \text{divisible by } 18$$

\therefore true for $n=1$

$$\text{assume true for } n=k \therefore 4^k + 6k + 8 = 18(k) \quad n \in \mathbb{Z}^+$$

$$n=k+1 = 4^{k+1} + 6(k+1) + 8$$

$$= 4(4^k) + 6k + 14$$

$$= 4[4^k + 6k + 8] - 18k - 18$$

$$= 4(18(k)) + 18(-k-1)$$

\downarrow
divisible by 18

\downarrow divisible by 18

$$\therefore 4^{k+1} + 6(k+1) + 8 \text{ is divisible by } 18$$

\therefore true for $n=1$ true for $n=k+1$ if true for $n=k$

\therefore by Mathematical induction true for all $n \in \mathbb{Z}^+$