1.
$$f(x) = (3 - 2x)^{-4}, \qquad |x| < \frac{3}{2}$$

Find the binomial expansion of f(x), in ascending powers of x, up to and including the term in x^2, giving each coefficient as a simplified fraction.

(4)

$$f(x) = \left[3\left(1 - \frac{2x}{3}\right)\right]^{-4} = \frac{1}{3^4}\left(1 - \frac{2x}{3}\right)^{-4}$$

$$\approx \frac{1}{81}\left[1 + (-4)\left(-\frac{2x}{3}\right) + (-4)(-5)\left(-\frac{2x}{3}\right)^2 \cdot \frac{1}{2}\right]$$

$$= \frac{1}{81}\left[1 + \frac{8x}{3} + \frac{40x^2}{9}\right]$$

$$= \frac{1}{81} + \frac{8x}{243} + \frac{40x^2}{729}$$

2. (a) Show that

$$\cot^2 x - \operatorname{cosec} x - 11 = 0$$

may be expressed in the form $\operatorname{cosec}^2 x - \operatorname{cosec} x + k = 0$, where k is a constant. **(1)**

(b) Hence solve for $0 \leqslant x < 360°$

$$\cot^2 x - \operatorname{cosec} x - 11 = 0$$

Give each solution in degrees to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.) **(5)**

a) $\cot^2 x \equiv \operatorname{cosec}^2 x - 1$

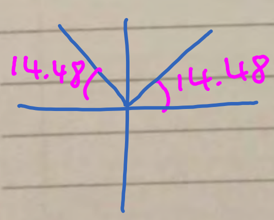
$\Rightarrow \operatorname{cosec}^2 x - 1 - \operatorname{cosec} x - 11 = 0$

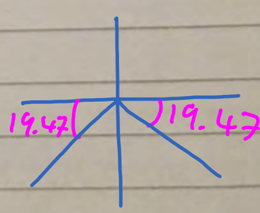
$\operatorname{cosec}^2 x - \operatorname{cosec} x - 12 = 0$

b) $(\operatorname{cosec} x - 4)(\operatorname{cosec} x + 3) = 0$

$\operatorname{cosec} x = 4$ or $\operatorname{cosec} x = -3$

$\sin x = \frac{1}{4}$ $\quad$ $\sin x = -\frac{1}{3}$





$\theta = 14.5°, 165.5°, 199.5°, 340.5°$

3. A curve C has equation

$$3^x + 6y = \frac{3}{2}xy^2$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates (2, 3). Give your answer in the form $\frac{a + \ln b}{8}$, where a and b are integers.

(7)

Let $z = 3^x$

$\ln z = x\ln 3$

$\frac{1}{z}\frac{dz}{dx} = \ln 3$

$\frac{dz}{dx} = 3^x \ln 3$

$\Rightarrow 3^x\ln 3 + 6\frac{dy}{dx} = \frac{3}{2}x \cdot 2y\frac{dy}{dx} + \frac{3}{2}y^2$

$\frac{dy}{dx} = \frac{3^x\ln 3 - \frac{3}{2}y^2}{3xy - 6}$

At (2, 3),

$\frac{dy}{dx} = \frac{3^2\ln 3 - \frac{3}{2}3^2}{3(2)(3) - 6} = \frac{\cancel{9}^{3}\ln 3 - \frac{\cancel{27}^{9}}{2}}{\cancel{12}^{4}}$

$= \frac{-9 + 6\ln 3}{8}$

$= \frac{-9 + \ln 729}{8}$

4.

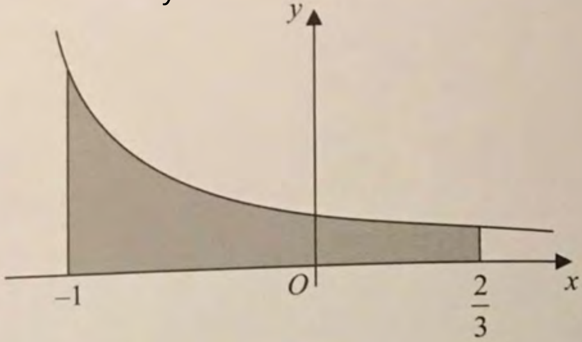


Figure 1

The curve C with equation $y = \frac{2}{(4+3x)}$, $x > -\frac{4}{3}$ is shown in Figure 1

The region bounded by the curve, the x-axis and the lines $x = -1$ and $x = \frac{2}{3}$, is shown shaded in Figure 1

This region is rotated through 360 degrees about the x-axis.

(a) Use calculus to find the exact value of the volume of the solid generated.

(5)

$$V = \pi \int_{-1}^{2/3} y^2 \, dx$$

$$= 4\pi \int (4+3x)^{-2} \, dx$$

$$= \frac{4\pi}{(-1)(3)} \left[\frac{1}{4+3x} \right]_{-1}^{2/3}$$

$$= \frac{-4\pi}{3} \left[\left(\frac{1}{6}\right) - (1) \right] = \frac{10\pi}{9}$$

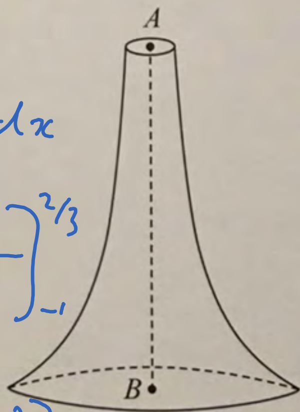


Figure 2

Figure 2 shows a candle with axis of symmetry AB where $AB = 15$ cm. A is a point at the centre of the top surface of the candle and B is a point at the centre of the base of the candle. The candle is geometrically similar to the solid generated in part (a).

(b) Find the volume of this candle.

(2)

$$\frac{V}{\frac{10\pi}{9}} = \left(\frac{15}{5/3}\right)^3$$

$$V = 810\pi$$

5.
$$f(x) = -x^3 + 4x^2 - 6$$

(a) Show that the equation $f(x) = 0$ has a root between $x = 1$ and $x = 2$

(2)

(b) Show that the equation $f(x) = 0$ can be rewritten as

$$x = \sqrt{\left(\frac{6}{4-x}\right)}$$

(2)

(c) Starting with $x_1 = 1.5$ use the iteration $x_{n+1} = \sqrt{\left(\frac{6}{4-x_n}\right)}$ to calculate the values of x_2, x_3 and x_4 giving all your answers to 4 decimal places.

(3)

(d) Using a suitable interval, show that 1.572 is a root of $f(x) = 0$ correct to 3 decimal places.

(2)

a) $f(1) = -(1)^3 + 4(1)^2 - 6 = -3 < 0$

$f(2) = -(2)^3 + 4(2)^2 - 6 = 2 > 0$

There's a change of sign ...

b) $0 = -x^3 + 4x^2 - 6$

$6 = x^2(4 - x)$

$x = \sqrt{\frac{6}{4-x}}$

c) $x_1 = 1.5$

$x_2 = 1.5492$

$x_3 = 1.5647$

$x_4 = 1.5696$

d) $f(1.5715) = -2.5 \times 10^{-3} < 0$

$f(1.5725) = 2.6 \times 10^{-3} > 0$

There's a sign change ...

6. A hot piece of metal is dropped into a cool liquid. As the metal cools, its temperature T degrees Celsius, t minutes after it enters the liquid, is modelled by

$$T = 300e^{-0.04t} + 20, \quad t \geqslant 0$$

(a) Find the temperature of the piece of metal as it enters the liquid.
(1)

(b) Find the value of t for which $T = 180$, giving your answer to 3 significant figures.

(Solutions based entirely on graphical or numerical methods are not acceptable.)
(4)

(c) Show, by differentiation, that the rate, in degrees Celsius per minute, at which the temperature of the metal is changing, is given by the expression

$$\frac{20 - T}{25}$$
(3)

a) At $t = 0$, $T = 300 + 20 = 320°C$

b) $180 = 300e^{-0.04t} + 20$

$8/15 = e^{-0.04t}$

$t = \frac{-\ln(8/15)}{0.04} = 15.7$ min. (3sf)

c) $\frac{dT}{dt} = -0.04(300)e^{-0.04t}$

$= -\frac{1}{25}(T - 20)$

$= \frac{20 - T}{25}$

7.

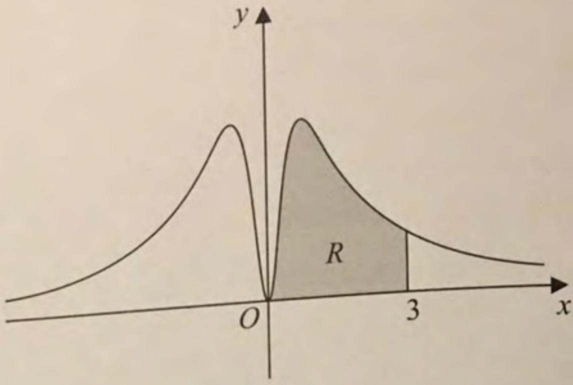


Figure 3

Figure 3 shows part of the curve C with equation

$$y = \frac{3\ln(x^2+1)}{(x^2+1)}, \quad x \in \mathbb{R}$$

(a) Find $\frac{dy}{dx}$ **(2)**

(b) Using your answer to (a), find the exact coordinates of the stationary point on the curve C for which $x > 0$. Write each coordinate in its simplest form. **(5)**

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the x-axis and the line $x = 3$

(c) Complete the table below with the value of y corresponding to $x = 1$

x	0	1	2	3
y	0	$\frac{3}{2}\ln 2$	$\frac{3}{5}\ln 5$	$\frac{3}{10}\ln 10$

(1)

(d) Use the trapezium rule with all the y values in the completed table to find an approximate value for the area of R, giving your answer to 4 significant figures. **(3)**

a) $\frac{dy}{dx} = \frac{3\cancel{(x^2+1)}\,2x\,\cancel{(x^2+1)}^{-1} - 3\ln(x^2+1)2x}{(x^2+1)^2}$

$= \frac{6x}{(x^2+1)^2}\left[1 - \ln(x^2+1)\right]$

b) $0 = \frac{6x}{(x^2+1)^2}\left[1 - \ln(x^2+1)\right]$

$x = 0$ or $1 - \ln(x^2+1) = 0$

$\ln(x^2+1) = 1$

$x^2 + 1 = e$

$x = \pm\sqrt{e-1}$

When $x = \sqrt{e-1}$, $y = \frac{3(1)}{e}$

$\therefore$ Stationary point is $\left(\sqrt{e-1}, 3e^{-1}\right)$

d) $R = \frac{1}{2}(1)\left[0 + \frac{3}{10}\ln 10 + 2\left(\frac{3}{2}\ln 2 + \frac{3}{5}\ln 5\right)\right]$

$= 2.351$ (4sf)

8.

$$f(\theta) = 9\cos^2\theta + \sin^2\theta$$

(a) Show that $f(\theta) = a + b\cos 2\theta$, where a and b are integers which should be found.

(3)

(b) Using your answer to part (a) and integration by parts, find the exact value of

$$\int_0^{\frac{\pi}{2}} \theta^2 f(\theta)\,d\theta$$

(6)

a) $\cos 2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$

$f(\theta) = 9\left(\dfrac{\cos 2\theta + 1}{2}\right) + \dfrac{1 - \cos 2\theta}{2}$

$= 5 + 4\cos 2\theta$

b) $u = \theta^2 \quad v' = f(\theta)$ | $u = \theta \quad v' = \sin 2\theta$

$u' = 2\theta \quad v = 5\theta + 2\sin 2\theta$ | $u' = 1 \quad v = -\frac{1}{2}\cos 2\theta$

$\int_0^{\pi/2} \theta^2 f(\theta)\,d\theta = \left[5\theta^3 + 2\theta\sin 2\theta\right]_0^{\pi/2} - \int_0^{\pi/2} 10\theta^2 + 4\theta\sin 2\theta\,d\theta$

$= \dfrac{5\pi^3}{8} - 10\left[\dfrac{\theta^3}{3}\right]_0^{\pi/2} - 4\int_0^{\pi/2} \theta\sin 2\theta\,d\theta$

$= \dfrac{5\pi^3}{24} + 2\left[\theta\cos 2\theta\right]_0^{\pi/2} - \int_0^{\pi/2} 2\cos 2\theta\,d\theta$

$= \dfrac{5\pi^3}{24} - \pi - \left[\sin 2\theta\right]_0^{\pi/2}$

$= \dfrac{5\pi^3}{24} - \pi$

9. (a) Express $\dfrac{3x^2 - 4}{x^2(3x-2)}$ in partial fractions.

(4)

(b) Given that $x > \dfrac{2}{3}$, find the general solution of the differential equation

$$x^2(3x-2)\frac{dy}{dx} = y(3x^2 - 4)$$

Give your answer in the form $y = f(x)$.

(6)

a) $\dfrac{3x^2-4}{x^2(3x-2)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{3x-2}$

$3x^2 - 4 \equiv Ax(3x-2) + B(3x-2) + Cx^2$

$x = 0: \quad -4 = -2B$

$B = 2$

Coef. x: $0 = -2A + 3B$

$A = 3$

Coef. x^2: $3 = 3A + C$

$C = -6$

$\Rightarrow \dfrac{3x^2-4}{x^2(3x-2)} = \dfrac{2}{x^2} + \dfrac{3}{x} - \dfrac{6}{3x-2}$

b) $\displaystyle\int \frac{1}{y}\,dy = \int \frac{3x^2-4}{x^2(3x-2)}\,dx$

$\ln y + C = -2x^{-1} + 3\ln|x| - 2\ln|3x-2|$

$\ln y = \ln\left(\dfrac{Ax^3}{3x-2}\right) - \dfrac{2}{x}$

$y = \dfrac{Ax^3}{3x-2} \cdot e^{-2/x}$

10. (a) Express $3\sin 2x + 5\cos 2x$ in the form $R\sin(2x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α to 3 significant figures.

(3)

(b) Solve, for $0 < x < \pi$,

$$3\sin 2x + 5\cos 2x = 4$$

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

$$g(x) = 4(3\sin 2x + 5\cos 2x)^2 + 3$$

(c) Using your answer to part (a) and showing your working,

(i) find the greatest value of $g(x)$,

(ii) find the least value of $g(x)$.

(4)

a) $R\sin(2x+\alpha) \equiv R\sin 2x\cos\alpha + R\cos 2x\sin\alpha$

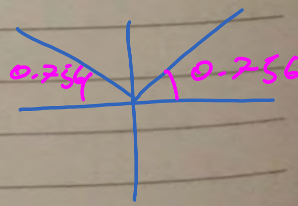
$\equiv 3\sin 2x + 5\cos 2x$

$5 = R\sin\alpha$ $\quad \tan\alpha = \frac{5}{3}$ $\quad R = \sqrt{5^2+3^2}$

$3 = R\cos\alpha$

$\alpha = 1.03$ $\quad = \sqrt{34}$

b) $\sqrt{34}\sin(2x+1.03) = 4$



$2x + 1.03 = 2.39, 7.04$

$x = 0.68, 3.01$

c) $g(x) = 4\underbrace{\left[\sqrt{34}\sin(2x+1.03)\right]^2}_{max = 34,\ min = 0} + 3$

i) $4(34)+3 = 139$ $\quad$ ii) $4(0)+3 = 3$

11.

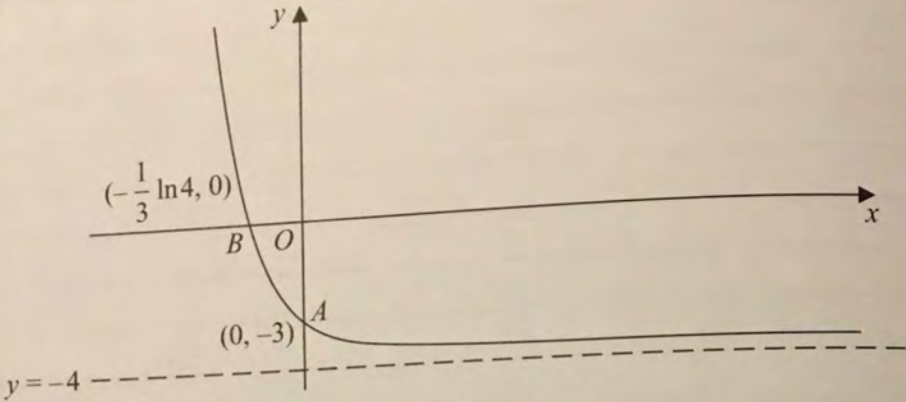


Figure 4

Figure 4 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, $x \in \mathbb{R}$

The curve meets the coordinate axes at the points $A(0, -3)$ and $B(-\frac{1}{3}\ln 4, 0)$ and the curve has an asymptote with equation $y = -4$

In separate diagrams, sketch the graph with equation

(a) $y = |\mathrm{f}(x)|$ **(4)**

(b) $y = 2\mathrm{f}(x) + 6$ **(3)**

On each sketch, give the exact coordinates of the points where the curve crosses or meets the coordinate axes and the equation of any asymptote.

Given that

$$\mathrm{f}(x) = \mathrm{e}^{-3x} - 4, \qquad x \in \mathbb{R}$$

$$\mathrm{g}(x) = \ln\left(\frac{1}{x+2}\right), \qquad x > -2$$

(c) state the range of f, **(1)**

$\mathrm{f}(x) > -4$

$y + 4 = e^{-3x}$

$x = -\frac{\ln(y+4)}{3}$

(d) find $\mathrm{f}^{-1}(x)$, **(3)**

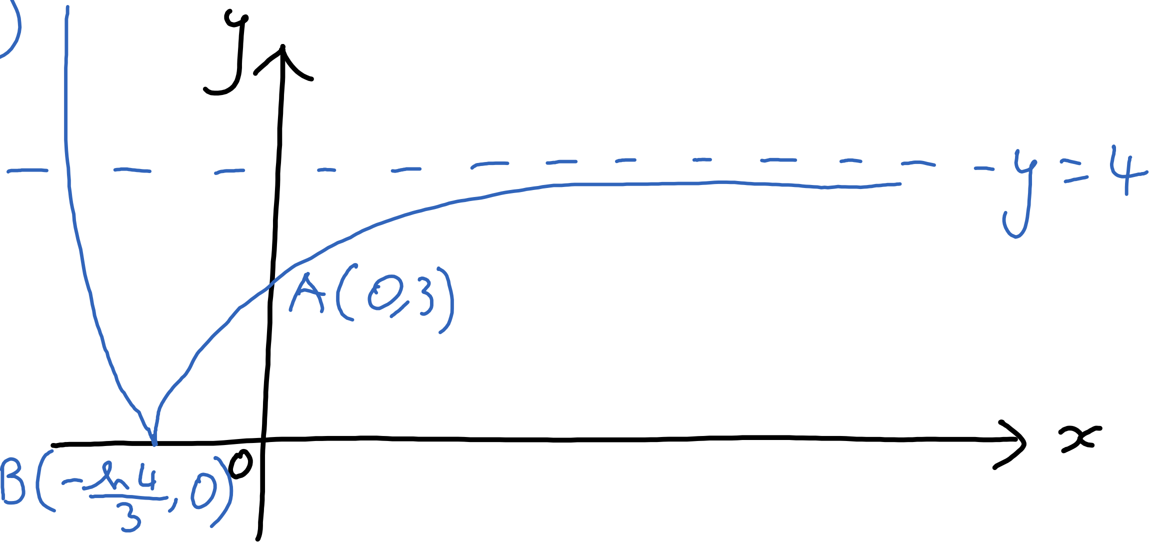
$f^{-1}(x) = -\frac{\ln(x+4)}{3}$

(e) express $\mathrm{fg}(x)$ as a polynomial in x. **(3)**

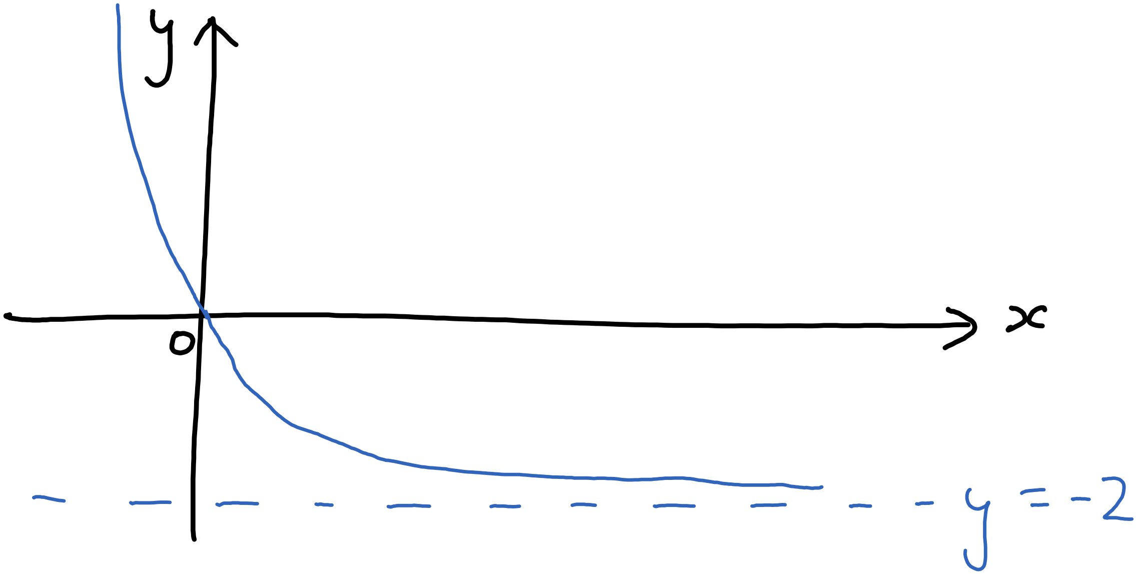
$fg(x) = e^{-3\ln\left(\frac{1}{x+2}\right)} - 4$

$= (x+2)^3 - 4$

a)



b)



12. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 12 \\ -4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Show that l_1 and l_2 meet, and find the position vector of their point of intersection A. **(6)**

(b) Find, to the nearest 0.1°, the acute angle between l_1 and l_2 **(3)**

The point B has position vector $\begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$.

(c) Show that B lies on l_1 **(1)**

(d) Find the shortest distance from B to the line l_2, giving your answer to 3 significant figures. **(4)**

a) $12 + 5\lambda = 2$ (1)

$-4 - 4\lambda = 2 + 6\mu$ (2)

$5 + 2\lambda = 3\mu$ (3)

(1): $\lambda = -2$

(2): $\mu = \dfrac{-6 - 4(-2)}{6} = 1/3$

(3): $5 + 2(-2) \stackrel{?}{=} 3(1/3)$

$5 - 4 = 1$

When $\mu = 1/3$, $\underline{r}_A = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$

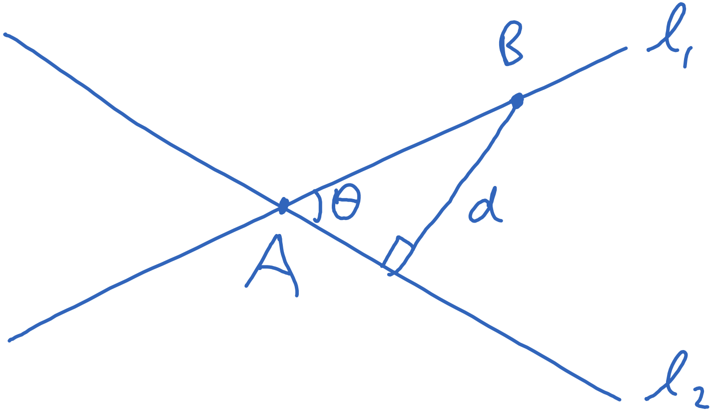
b) $\cos\theta = \left| \dfrac{5(0) - 4(6) + 2(3)}{\sqrt{5^2 + 4^2 + 2^2}\sqrt{6^2 + 3^2}} \right|$

$\theta = 66.4°$

c) When $\lambda = -1$,

$\underline{r} = \begin{pmatrix} 12 - 5 \\ -4 + 4 \\ 5 - 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$

d)



$$d = AB \sin\theta$$

$$= \sqrt{(7-2)^2 + (4-0)^2 + (3-1)^2} \sin 66.4$$

$$= 6.15 \ (3sf)$$

13. A curve C has parametric equations

$$x = 6\cos 2t, \qquad y = 2\sin t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

(a) Show that $\frac{dy}{dx} = \lambda \operatorname{cosec} t$, giving the exact value of the constant λ.

(4)

(b) Find an equation of the normal to C at the point where $t = \frac{\pi}{3}$

Give your answer in the form $y = mx + c$, where m and c are simplified surds.

(6)

The cartesian equation for the curve C can be written in the form

$$x = f(y), \qquad -k < y < k$$

where $f(y)$ is a polynomial in y and k is a constant.

(c) Find $f(y)$.

(3)

(d) State the value of k.

(1)

a) $\frac{dx}{dt} = -12\sin 2t \qquad \frac{dy}{dt} = 2\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2\cos t}{-12\sin 2t} = \frac{2\cos t}{-24\sin t\cos t}$

$= -\frac{1}{12} \operatorname{cosec} t$

b) At $t = \frac{\pi}{3}$, $x = 6\cos(2\pi/3) = -3$

$y = 2\sin(\pi/3) = \sqrt{3}$

$\frac{dy}{dx} = -\frac{1}{12} \cdot \frac{1}{\sin \pi/3} = -\frac{1}{6\sqrt{3}}$

$\Rightarrow m_n = 6\sqrt{3}$

$y - y_1 = m(x - x_1)$

$y - \sqrt{3} = 6\sqrt{3}(x+3)$

$y = 6\sqrt{3}x + 19\sqrt{3}$

c) $x = 6\cos 2t = 6(1 - 2\sin^2 t)$

$= 6 - 12\sin^2 t$

$= 6 - 3y^2$

d) $k = 2$