1. A sequence of numbers $u_1, u_2, u_3, \ldots$ satisfies

$$u_{n+1} = 2u_n - 6, \quad n \geqslant 1$$

Given that $u_1 = 2$

(a) find the value of u_3

(2)

(b) evaluate $\sum_{i=1}^{4} u_i$

(3)

a) $u_2 = 2(2) - 6 = -2$

$u_3 = 2(-2) - 6 = -10$

b) $\sum_{i=1}^{4} u_i = u_1 + u_2 + u_3 + u_4$

$= 2 + (-2) + (-10) + [2(-10) - 6]$

$= -36$

2. (i) Given that $\dfrac{49}{\sqrt{7}} = 7^a$, find the value of a.

(2)

(ii) Show that $\dfrac{10}{\sqrt{18} - 4} = 15\sqrt{2} + 20$

You must show all stages of your working.

(3)

i) $7^a = \dfrac{49}{\sqrt{7}} = \dfrac{7^2}{7^{1/2}} = 7^{3/2}$

$\Rightarrow a = \dfrac{3}{2}$

ii) $\dfrac{10}{\sqrt{18} - 4} = \dfrac{10}{\sqrt{18} - 4} \times \dfrac{-\sqrt{18} - 4}{-\sqrt{18} - 4}$

$= \dfrac{-10\sqrt{18} - 40}{16 - 18}$

$= 5\sqrt{18} + 20$

3. Find, using calculus and showing each step of your working,

$$\int_1^4 \left(6x - 3 - \frac{2}{\sqrt{x}}\right) dx$$

(5)

$$\int_1^4 \left(6x - 3 - 2x^{-1/2}\right) dx = \left[3x^2 - 3x - 4x^{1/2}\right]_1^4$$

$$= \left[3(4)^2 - 3(4) - 4(4)^{1/2}\right] - \left[3(1)^2 - 3(1) - 4(1)^{1/2}\right]$$

$$= 48 - 12 - 8 - 3 + 3 + 4$$

$$= 32$$

4. The 4th term of an arithmetic sequence is 3 and the sum of the first 6 terms is 27

Find the first term and the common difference of this sequence.

(6)

$u_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$3 = a + (4-1)d$

$a = 3 - 3d \quad (1)$

$27 = \frac{6}{2}[2a + (6-1)d]$

$27 = 6a + 15d \quad (2)$

(1) in (2): $27 = 6(3-3d) + 15d$

$= 18 - 3d$

$d = -3$

In (1): $a = 3 - 3(-3)$

$= 12$

5. (a) Sketch the graph of $y = \sin 2x$, $\quad 0 \leqslant x \leqslant \frac{3\pi}{2}$

Show the coordinates of the points where your graph crosses the x-axis.

(2)

The table below gives corresponding values of x and y, for $y = \sin 2x$.
The values of y are rounded to 3 decimal places where necessary.

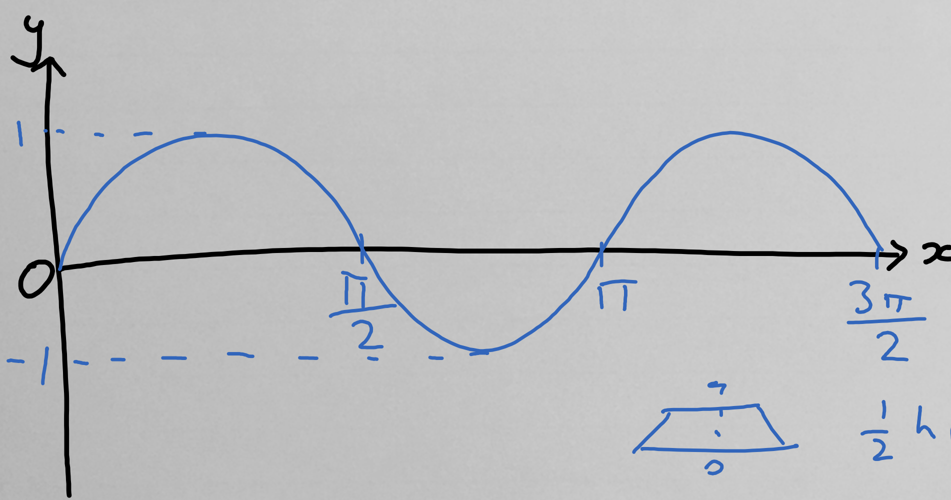
x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
y	0	0.5	0.866	1

(b) Use the trapezium rule with all the values of y from the table to find an approximate value for

$$\int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

(3)

a)



$\frac{1}{2}h(a+b)$

b) $\int_0^{\pi/4} \sin 2x \, dx \approx \frac{1}{2} \cdot \frac{\pi}{12}\left[0+1+2(0.5+0.866)\right]$

$= 0.489$ (3dp)

6.

$$f(x) = x^3 + x^2 - 12x - 18$$

(a) Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

(2)

(b) Factorise $f(x)$.

(2)

(c) Hence find exact values for all the solutions of the equation $f(x) = 0$

(3)

a) $f(-3) = (-3)^3 + (-3)^2 - 12(-3) - 18$

$= -27 + 9 + 36 - 18$

$= 0$

b) $f(x) = x^3 + x^2 - 12x - 18$

$= (x+3)(x^2 - 2x - 6)$

c) $f(x) = 0$

$x = -3$ or $x = \dfrac{2 \pm \sqrt{(-2)^2 + 4(6)}}{2}$

$= 1 \pm \sqrt{7}$

7. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 + kx)^8$, where k is a non-zero constant. Give each term in its simplest form.

(4)

Given that the coefficient of x^3 in this expansion is 1512

(b) find the value of k.

(3)

a) $(1+kx)^8 \approx 1 + 8kx + 8(7)(kx)^2\frac{1}{2}$

$+ 8(7)(6)(kx)^3\frac{1}{3!}$

$= 1 + 8kx + 28k^2x^2 + 56k^3x^3$

b) $1512 = 56k^3$

$k^3 = 27$

$k = 3$

8. (a) Given that $7\sin x = 3\cos x$, find the exact value of $\tan x$.

(1)

(b) Hence solve for $0 \leqslant \theta < 360°$

$$7\sin(2\theta + 30°) = 3\cos(2\theta + 30°)$$

giving your answers to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

a) $7\sin x = 3\cos x$

$\tan x = \frac{3}{7}$

b) $7\sin(2\theta + 30) = 3\cos(2\theta + 30)$

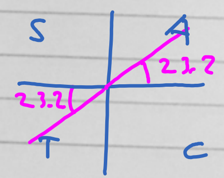
$\tan(2\theta + 30) = \frac{3}{7}$ $\quad 0 \leqslant \theta < 360$

$30 \leqslant 2\theta + 30 < 750$

$2\theta + 30 = \cancel{23.2}, 203.2,$

$383.2, 563.2,$

743.2



$\theta = 86.6, 176.6, 266.6, 356.6$

9. The resident population of a city is 130 000 at the end of Year 1

A model predicts that the resident population of the city will increase by 2% each year, with the populations at the end of each year forming a geometric sequence.

(a) Show that the predicted resident population at the end of Year 2 is 132 600

(1)

(b) Write down the value of the common ratio of the geometric sequence.

(1)

The model predicts that Year N will be the first year which will end with the resident population of the city exceeding 260 000

(c) Show that

$$N > \frac{\log_{10} 2}{\log_{10} 1.02} + 1$$

(4)

(d) Find the value of N.

(1)

a) $P_2 = 1.02 P_1 = 1.02 \times 130000$

$= 132600$

b) 1.02

c) $P_N = P_1 (1.02)^{N-1}$

$130000(1.02)^{N-1} > 260000$

$1.02^{N-1} > 2$

$\log_{10} 1.02^{N-1} > \log_{10} 2$

$(N-1)\log_{10} 1.02 > \log_{10} 2$

$N > \dfrac{\log_{10} 2}{\log_{10} 1.02} + 1$

10. The curve C has equation

$$y = 12x^{\frac{5}{4}} - \frac{5}{18}x^2 - 1000, \qquad x > 0$$

(a) Find $\frac{dy}{dx}$

(2)

(b) Hence find the coordinates of the stationary point on C.

(5)

(c) Use $\frac{d^2y}{dx^2}$ to determine the nature of this stationary point.

(3)

a) $\frac{dy}{dx} = 15x^{1/4} - \frac{5x}{9}$

b) $0 = 15x^{1/4} - \frac{5x}{9}$

$= 27x^{1/4} - x$

$= x^{1/4}(27 - x^{3/4})$

$x = 0$ or $x^{3/4} = 27$

But $x > 0$ $\quad x = 27^{4/3}$

$= 81$

When $x = 81$, $y = 12(81)^{5/4} - \frac{5}{18}(81)^2 - 1000$

$= 93.5$

$\therefore$ Stationary point at $(81, 93.5)$

c) $\frac{d^2y}{dx^2} = \frac{15}{4}x^{-3/4} - \frac{5}{9}$

When $x = 81$,

$\frac{d^2y}{dx^2} = \frac{1}{4} \cdot \frac{15}{81^{3/4}} - \frac{5}{9}$

$= -\frac{5}{12} < 0$

$\therefore$ Max pt.

11.

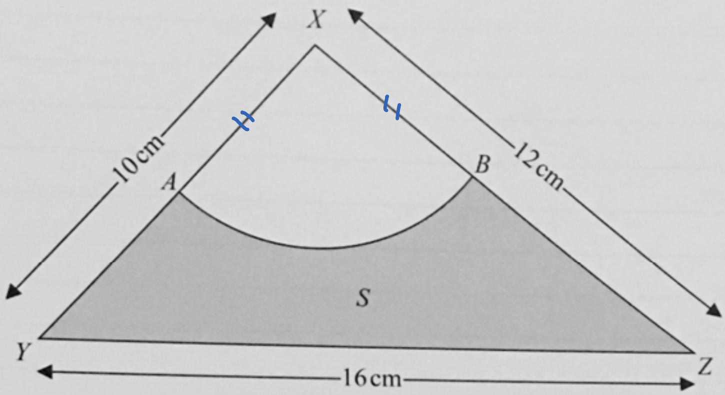


Figure 1

Figure 1 shows a triangle XYZ with $XY = 10$ cm, $YZ = 16$ cm and $ZX = 12$ cm.

(a) Find the size of the angle YXZ, giving your answer in radians to 3 significant figures.

(3)

The point A lies on the line XY and the point B lies on the line XZ and $AX = BX = 5$ cm. AB is the arc of a circle with centre X.

The shaded region S, shown in Figure 1, is bounded by the lines BZ, ZY, YA and the arc AB.

Find

(b) the perimeter of the shaded region to 3 significant figures,

(4)

(c) the area of the shaded region to 3 significant figures.

(4)

a) $\cos \widehat{YXZ} = \dfrac{12^2 + 10^2 - 16^2}{2 \times 10 \times 12}$

$\widehat{YXZ} = 1.62^c$ (3sf)

b) $\widehat{AB} = 5 \times 1.62 = 8.10$ cm

$P = 8.10 + (10-5) + (12-5) + 16 = 36.1$ cm

c) Area $XYZ = \frac{1}{2} \times 10 \times 12 \sin 1.62 = 59.9$ cm^2

Area of sector $= \frac{1}{2} \times 5^2 \times 1.62 = 20.3$ cm^2

$S = 59.9 - 20.3 = 39.7$ cm^2

12.

$$f(x) = \frac{(4+3\sqrt{x})^2}{x}, \qquad x > 0$$

(a) Show that $f(x) = Ax^{-1} + Bx^k + C$, where A, B, C and k are constants to be determined.

(4)

(b) Hence find $f'(x)$.

(2)

(c) Find an equation of the tangent to the curve $y = f(x)$ at the point where $x = 4$

(4)

a) $f(x) = \dfrac{16 + 24\sqrt{x} + 9x}{x} = 16x^{-1} + 24x^{-1/2} + 9$

b) $f'(x) = -16x^{-2} - 12x^{-3/2}$

c) $f'(4) = -\dfrac{16}{4^2} - \dfrac{12}{4^{3/2}} = -1 - \dfrac{3}{2} = -\dfrac{5}{2}$

$f(4) = \dfrac{(4+3\sqrt{4})^2}{4} = 25$

$y - y_1 = m(x - x_1)$

$y - 25 = -\dfrac{5}{2}(x - 4)$

$y = 35 - \dfrac{5}{2}x$

13. The equation $k(3x^2 + 8x + 9) = 2 - 6x$, where k is a real constant, has no real roots.

(a) Show that k satisfies the inequality

$$11k^2 - 30k - 9 > 0$$

(4)

(b) Find the range of possible values for k.

(4)

a) $k(3x^2 + 8x + 9) = 2 - 6x$

$3kx^2 + 8kx + 9k = 2 - 6x$

$3kx^2 + (8k+6)x + 9k - 2 = 0$

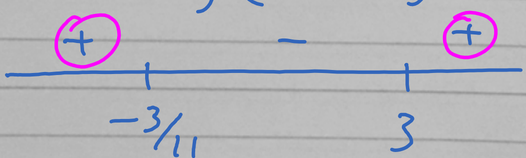
$\Delta = (8k+6)^2 - 4(3k)(9k-2) < 0$

$64k^2 + 96k + 36 - 108k^2 + 24k < 0$

$44k^2 - 120k - 36 > 0$

$11k^2 - 30k - 9 > 0$

b) $(11k + 3)(k - 3) > 0$



$\{k < -\frac{3}{11}\} \cup \{k > 3\}$

14. (i) Given that

$$\log_a x + \log_a 3 = \log_a 27 - 1, \text{ where } a \text{ is a positive constant}$$

find, in its simplest form, an expression for x in terms of a.

(4)

(ii) Solve the equation

$$(\log_5 y)^2 - 7(\log_5 y) + 12 = 0$$

showing each step of your working.

(4)

i) $\log x + \log 3 = \log 27 - 1$

$$\log_a\left(\frac{3x}{27}\right) = -1$$

$$\frac{x}{9} = a^{-1}$$

$$x = \frac{9}{a}$$

ii) $(\log_5 y)^2 - 7(\log_5 y) + 12 = 0$

Let $x = \log_5 y$:

$$x^2 - 7x + 12 = 0$$

$$(x-4)(x-3) = 0$$

$\log_5 y = 4$ or $\log_5 y = 3$

$y = 5^4$ $\quad$ $y = 5^3$

$= 625$ $\quad$ $= 125$

15. The points A and B have coordinates $(-8, -8)$ and $(12, 2)$ respectively. AB is the diameter of a circle C.

(a) Find an equation for the circle C.

The point $(4, 8)$ also lies on C.

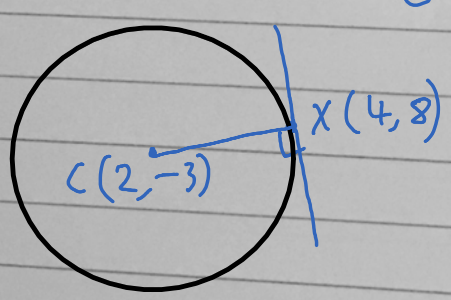
(b) Find an equation of the tangent to C at the point $(4, 8)$, giving your answer in the form $ax + by + c = 0$

a) Centre is mid-pt of AB: $C(2, -3)$

Radius, $r^2 = (12-2)^2 + (2+3)^2 = 125$

$\Rightarrow (x-2)^2 + (y+3)^2 = 125$

b)



$m_{cx} = \dfrac{8+3}{4-2} = \dfrac{11}{2}$

$m_{tangent} = -\dfrac{2}{11}$

$y - y_1 = m(x - x_1)$

$y - 8 = -\dfrac{2}{11}(x-4)$

$2x + 11y - 96 = 0$

16.

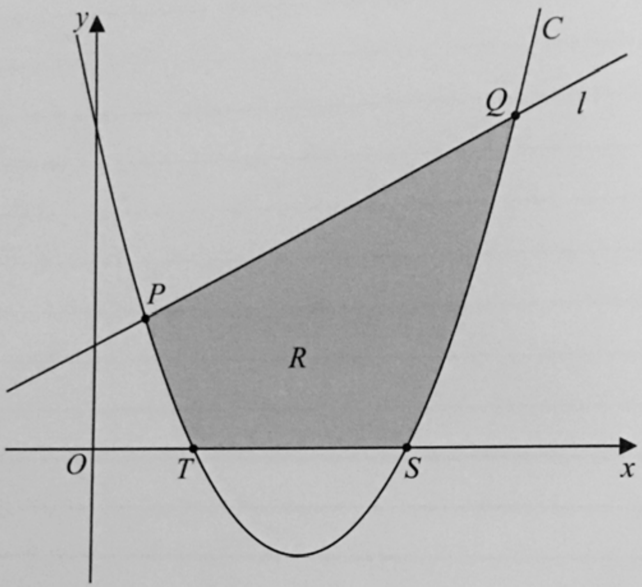


Figure 2

The straight line l with equation $y = \frac{1}{2}x + 1$ cuts the curve C, with equation $y = x^2 - 4x + 3$, at the points P and Q, as shown in Figure 2

(a) Use algebra to find the coordinates of the points P and Q.

(5)

The curve C crosses the x-axis at the points T and S.

(b) Write down the coordinates of the points T and S.

(2)

The finite region R is shown shaded in Figure 2. This region R is bounded by the line segment PQ, the line segment TS, and the arcs PT and SQ of the curve.

(c) Use integration to find the exact area of the shaded region R.

(8)

a) $\frac{1}{2}x + 1 = x^2 - 4x + 3$

$0 = 2x^2 - 9x + 4$

$= (2x-1)(x-4)$

$x = \frac{1}{2}$ or 4

when $x = \frac{1}{2}$, $y = \frac{1}{2}\left(\frac{1}{2}\right) + 1 = \frac{5}{4}$

$x = 4$, $y = \frac{1}{2}(4) + 1 = 3$

$\therefore P\left(\frac{1}{2}, \frac{5}{4}\right)$ and $Q(4, 3)$

b) $y = x^2 - 4x + 3$

$= (x-3)(x-1)$

$\therefore T(1,0)$ and $S(3,0)$

c) $R = \int_{1/2}^{4} \left[\frac{x}{2} + 1 - (x^2 - 4x + 3)\right] dx - \left| \int_1^3 (x^2 - 4x + 3)\, dx \right|$

$= \int_{1/2}^{4} \left(-x^2 + \frac{9x}{2} - 2\right) dx + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3$

$= \left[-\frac{x^3}{3} + \frac{9x^2}{4} - 2x\right]_{1/2}^{4} + 9 - 18 + 9 - \frac{1}{3} + 2 - 3$

$= -\frac{64}{3} + 36 - 8 + \frac{1}{24} - \frac{9}{16} + 1 - \frac{4}{3}$

$= \frac{93}{16}$