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1.

$$f(x) = x^4 - x^3 - 9x^2 + 29x - 60$$

Given that $x = 1 + 2i$ is a root of the equation $f(x) = 0$, use algebra to find the three other roots of the equation $f(x) = 0$

(7)

$x = 1 \pm 2i$

$x - 1 = \pm 2i$

$(x-1)^2 = -4$

$x^2 - 2x + 5 = 0$

$$\begin{array}{r l}
 & x^2 + x - 12 \\
x^2 - 2x + 5 \; \big) & x^4 - x^3 - 9x^2 + 29x - 60 \\
 & -(x^4 - 2x^3 + 5x^2) \\
 & x^3 - 14x^2 + 29x \\
 & -(x^3 - 2x^2 + 5x) \\
 & -12x^2 + 24x - 60 \\
 & -(-12x^2 + 24x - 60) \\
 & 0
\end{array}$$

$(x^2 - 2x + 5)(x^2 + x - 12) = x^4 - x^3 - 9x^2 + 29x - 60$

$x = 1 \pm 2i$ or $x = 3$ & -4

$x = 3$

$x = -4$

$x = 1 - 2i$

Leave blank

2.

$$f(x) = x^3 - 3x^2 + \frac{1}{2\sqrt{x^5}} + 2, \quad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [2,3].

(2)

(b) Taking 3 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places.

(5)

a) $f(2) = -1.9116...$
$f(3) = 2.032...$

- change of sign
- $f(x)$ is continuous
- root α exists between $x = 2$ & $x = 3$.

b) $x_n - \frac{f(x_n)}{f'(x_n)}$

$f'(x) = 3x^2 - 6x - \frac{5}{4}x^{-3.5}$ when $x = 3$ $f'(x) = 8.973...,$

$3 - \frac{2.032075015}{8.973270821}$

$\alpha = 2.774$

Leave blank

3. Given that $z = x + \mathrm{i}y$, where x and y are real numbers, solve the equation

$$(z - 2\mathrm{i})(z^* - 2\mathrm{i}) = 21 - 12\mathrm{i}$$

where z^* is the complex conjugate of z.

(6)

$z^* = x - iy$

$(x + iy - 2i)(x - iy - 2i)$

$x^2 - xyi - 2xi + xyi - 2xi + y^2 - 4$

$x^2 + y^2 - 4xi - 4$

$(x^2 + y^2 - 4) + i(-4x)$

$u + iv$

$x^2 + y^2 - 4 = 21$ $\quad$ $-4x = -12$

$4x = 12$

$x = 3$

$(3)^2 + y^2 - 4 = 21$

$9 + y^2 - 4 = 21$

$y^2 = 16$ $\quad y = \pm 4$

$z = 3 \pm 4i$

Leave blank

4. The parabola C has cartesian equation $y^2 = 12x$

The point $P(3p^2, 6p)$ lies on C, where $p \neq 0$

(a) Show that the equation of the normal to the curve C at the point P is

$$y + px = 6p + 3p^3$$

(5)

This normal crosses the curve C again at the point Q.

Given that $p = 2$ and that S is the focus of the parabola, find

(b) the coordinates of the point Q,

(5)

(c) the area of the triangle PQS.

(4)

a) $y = \sqrt{12}x^{\frac{1}{2}} = 2\sqrt{3}x^{\frac{1}{2}}$

$\frac{dy}{dx} = \sqrt{3}x^{-\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{x}}$ when $x = 3p^2$ $\frac{\sqrt{3}}{\sqrt{3p^2}} = \frac{1}{p} = m$

at the normal $m_n = -p$

$y - y_1 = m(x - x_1)$

$y - 6p = -p(x - 3p^2)$

$y + px = 6p + 3p^3$

b) $p = 2$ $\quad y + 2x = 12 + 24$ $\quad y^2 = 12x$

$y + 2x = 36$ $\quad \frac{y^2}{12} = x$ $\quad \frac{y^2}{6} = 2x$

$y \cancel{*} + \frac{y^2}{6} = 36$

$\cancel{\frac{6y+y^2}{6}}$ $\quad 6y + y^2 = 216$

$y^2 + 6y - 216 = 0$ $\quad (y+18)(y-12) = 0$

$y = -18$ or 27

c) Focus is $(3, 0)$

$y = 0$ $\quad x = 18$

Area $= \frac{1}{2} \times (18-3) \times 12 + \frac{1}{2}(18-3)(18) = 225$

Leave blank

5. The quadratic equation

$$4x^2 + 3x + 1 = 0$$

has roots α and β.

(a) Write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$.

(2)

(b) Find the value of $(\alpha^2 + \beta^2)$.

(2)

(c) Find a quadratic equation which has roots

$$(4\alpha - \beta) \text{ and } (4\beta - \alpha)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers to be determined.

(4)

5) a) $\alpha + \beta = -\frac{3}{4}$ $\quad \alpha\beta = \frac{1}{4}$

b) $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\left(-\frac{3}{4}\right)^2 - 2\left(\frac{1}{4}\right) = \frac{1}{16}$

c) sum $(4\alpha - \beta) + (4\beta - \alpha) = 3(\alpha + \beta) = 3\left(-\frac{3}{4}\right)$

$= -\frac{9}{4}$

product $(4\alpha - \beta)(4\beta - \alpha)$

$16\alpha\beta - 4\beta^2 - 4\alpha^2 + \alpha\beta \;= 17\alpha\beta - 4(\alpha^2 + \beta^2)$

$17\left(\frac{1}{4}\right) - 4\left(\frac{1}{16}\right) = 4$

$x^2 - \left(-\frac{9}{4}x\right) + 4 = 0$

$4x^2 + 9x + 16 = 0$

Leave blank

6.

(i) $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

(a) Describe fully the single transformation represented by the matrix **A**.

(2)

(b) Describe fully the single transformation represented by the matrix **B**.

(2)

The transformation represented by **A** followed by the transformation represented by **B** is equivalent to the transformation represented by the matrix **C**.

(c) Find **C**.

(2)

(ii) $\mathbf{M} = \begin{pmatrix} 2k+5 & -4 \\ 1 & k \end{pmatrix}$, where k is a real number.

Show that det $\mathbf{M} \neq 0$ for all values of k.

(4)

i)a) stretch scale factor 3 parallel to x axis

b) rotation 210 degrees anticlockwise about (0,0).

c) $C = BA = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -\frac{3\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

ii) det $M = \cancel{k}(2k+5)k - 1(-4)$

$2k^2 + 5k + 4$

$b^2 - 4ac$

$25 - 32$

$b^2 - 4ac < 0$

no real roots

so det $M \neq 0$

Leave blank

7. Given that, for all positive integers n,

$$\sum_{r=1}^{n}(r+a)(r+b)=\frac{1}{6}n(2n+11)(n-1)$$

where a and b are constants and $a>b$,

(a) find the value of a and the value of b.

(8)

(b) Find the value of

$$\sum_{r=9}^{20}(r+a)(r+b)$$

(3)

7)a) $(r+a)(r+b)$ $r^2+ra+rb+ab$

$$\sum r^2+\sum r(a+b)+\sum ab$$

$$\frac{n}{6}(n+1)(2n+1)+\frac{n}{2}(n+1)(a+b)+abn=\frac{n}{6}(2n+11)(n-1)$$

$$\cancel{\frac{n}{6}}\left[(n+1)(2n+1)+3(n+1)(a+b)+6ab\right]=\cancel{\frac{n}{6}}(2n+11)(n-1)$$

$$\cancel{2n^2}+3n+1+3(n+1)(a+b)+6ab=\cancel{2n^2}+9n-11$$

$$3n+1+3na+3a+3nb+3b+6ab=9n-11$$

$$n(3+3a+3b)+(3a+3b+1+6ab)=9n-11$$

$n(1+a+b)=3n$ $\qquad 3a+3b+1+6ab=-11$

$1+a+b=3$ $\qquad 3(2)+1+6ab=-11$

$a+b=2$ $\qquad ab=-3$

$a=3$ $\qquad b=-1$

b) $\sum_{r=9}^{20}(r+a)(r+b)$

$f(20)-f(8)$

$$\frac{1}{6}(20)(51)(19)-\frac{1}{6}(8)(27)(7)$$

$$=2978$$

Leave blank

8. (i) A sequence of numbers is defined by

$$u_1 = 5 \qquad u_2 = 13$$

$$u_{n+2} = 5u_{n+1} - 6u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 2^n + 3^n$$

(6)

(ii) Prove by induction that for $n \geqslant 2$, where $n \in \mathbb{Z}$,

$$f(n) = 7^{2n} - 48n - 1$$

is divisible by 2304

(6)

8) i) $n=1 \quad u_1 = 2^1 + 3^1 = 5$ true for $n=1$ & $n=2$
$n=2 \quad u_2 = 2^2 + 3^2 = 13$

assume $u_k = 2^k + 3^k \quad u_{k+1} = 2^{k+1} + 3^{k+1}$

$u_{k+2} = 5u_{k+1} - 6u_k = 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k)$

$5(2^{k+1}) + 5(3^{k+1}) - 6(2^k) - 6(3^k)$
$5(2^{k+1}) + 5(3^{k+1}) - 3(2)(2^k) - 3(2)(3^k)$

$5(2^{k+1}) + 5(3^{k+1}) - 3(2^{k+1}) - 2(3^{k+1})$
$= 2(2^{k+1}) + 3(3^{k+1})$
$2^{(k+1)+1} + 3^{(k+1)+1} = 2^{k+2} + 3^{k+2}$

if true for k & k+1 then shown true k+2
& as true for n=1 & n=2, true $n \in \mathbb{Z}^+$

ii) $f(2) = 7^4 - 48(2) - 1 = 2304$ so true $n=2$
Assume $7^{2k} - 48k - 1 = 2304p$ interger p.
$f(k+1) - f(k) = 7^{2k+2} - 48(k+1) - 1 - (7^{2k} - 48k - 1)$
$7^{2k+2} - 7^{2k} - 48 = 7^{2k}(49-1) - 48$
$48 f(k) + 48^2 k$
$48(2304p) + 2304k \quad f(k+1) = 49(2304p) + 2304k$

if true k, shown true k+1 and as true for n=2
true for $n \geqslant 2 \quad n \in \mathbb{Z}$