1 Find the set of positive integers n for which n does not divide $(n-1)!$. Justify your answer. [Note that small values of n may require special consideration.]

2 Let $I_{m,n} = \int \cos^m x \sin nx \, \mathrm{d}x$, where m and n are non-negative integers. Prove that for $m, n \geqslant 1$,

$$(m+n)\, I_{m,n} = -\cos^m x \cos nx + m I_{m-1,n-1}\,.$$

(i) Show that $\displaystyle\int_0^{\pi} \cos^m x \sin nx \, \mathrm{d}x = 0$ whenever m, n are both even or both odd.

(ii) Evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^2 x \sin 3x \, \mathrm{d}x$.

3 (*a*) If $z = x + \mathrm{i}y$, with x, y real, show that

$$|x| \cos\alpha + |y| \sin\alpha \leqslant |z| \leqslant |x| + |y|$$

for all real α.

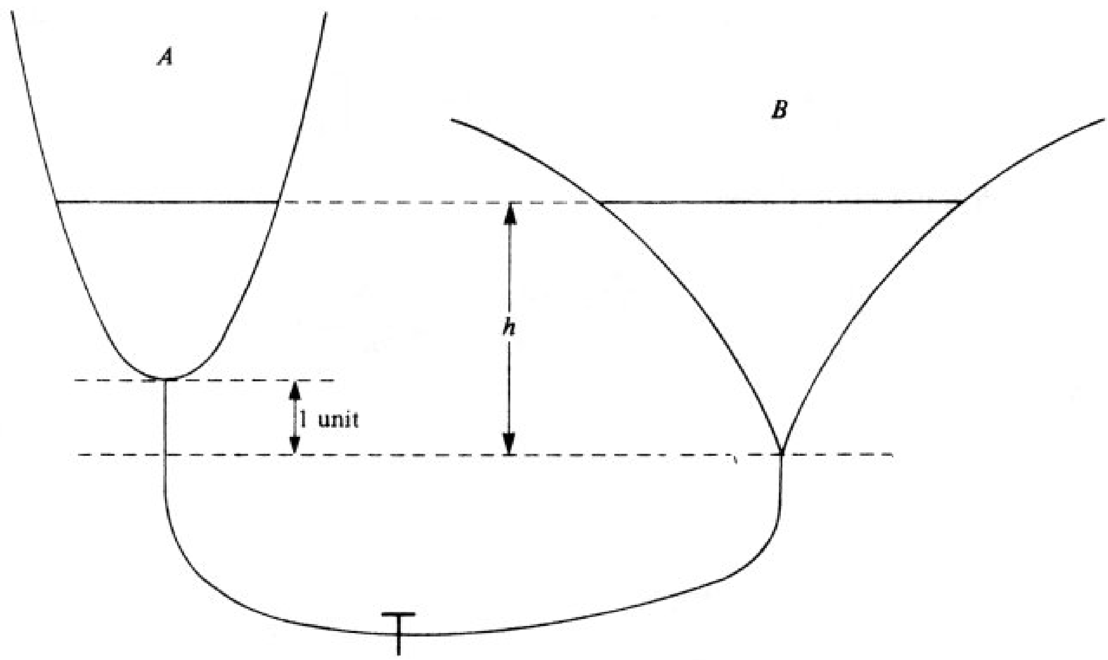
(*b*) By considering $(5-\mathrm{i})^4(1+\mathrm{i})$, show that

$$\tfrac{1}{4}\pi = 4\tan^{-1}(\tfrac{1}{5}) - \tan^{-1}(\tfrac{1}{239}).$$

Prove similarly that

$$\tfrac{1}{4}\pi = 3\tan^{-1}(\tfrac{1}{4}) + \tan^{-1}(\tfrac{1}{20}) + \tan^{-1}(\tfrac{1}{1985}).$$

4



Two funnels A and B have surfaces formed by rotating the curves $y = x^2$ and $y = 2\sinh^{-1}x$ $(x > 0)$ about the y-axis. The bottom of B is one unit lower than the bottom of A and they are connected by a thin rubber tube with a tap in it. The tap is closed and A is filled with water to a depth of 4 units. The tap is then opened. When the water comes to rest, both surfaces are at a height h above the bottom of B, as shown in the diagram. Show that h satisfies the equation

$$h^2 - 3h + \sinh h = 15.$$

5 A secret message consists of the numbers 1, 3, 7, 23, 24, 37, 39, 43, 43, 43, 45, 47 arranged in some order as $a_1, a_2, \ldots, a_{12}$. The message is encoded as $b_1, b_2, \ldots, b_{12}$, with $0 \leqslant b_j \leqslant 49$ and

$$b_{2j} \equiv a_{2j} + n_0 + j \pmod{50},$$
$$b_{2j+1} \equiv a_{2j+1} + n_1 + j \pmod{50},$$

for some integers n_0 and n_1. If the coded message is 35, 27, 2, 36, 15, 35, 8, 40, 40, 37, 24, 48, find the original message, explaining your method carefully.

6 The functions $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$\frac{dx}{dt} + 2x - 5y = 0,$$

$$\frac{dy}{dt} + ax - 2y = 2\cos t,$$

subject to $x = 0, \dfrac{dy}{dt} = 0$ at $t = 0$.

Solve these equations for x and y in the case when $a = 1$.

Without solving the equations explicitly, state briefly how the form of the solutions for x and y if $a > 1$ would differ from the form when $a = 1$.

7 Prove that

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots + \frac{(-1)^n t^{2n+1}}{2n+1} + (-1)^{n+1}\int_0^t \frac{x^{2n+2}}{1+x^2}\,\mathrm{d}x.$$

Hence show that, if $0 \leqslant t \leqslant 1$, then

$$\frac{t^{2n+3}}{2(2n+3)} \leqslant \left| \tan^{-1} t - \sum_{r=0}^{n} \frac{(-1)^r t^{2r+1}}{2r+1} \right| \leqslant \frac{t^{2n+3}}{2n+3}.$$

Show that, as $n \to \infty$,

$$4\sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)} \to \pi,$$

but that the error in approximating π by $4\displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)}$ is at least 10^{-2} if n is less than or equal to 98.

8 Show that, if the lengths of the diagonals of a parallelogram are specified, then the parallelogram has maximum area when the diagonals are perpendicular. Show also that the area of a parallelogram is less than or equal to half the square of the length of its longer diagonal.

The set A of points (x, y) is given by

$$|a_1x + b_1y - c_1| \leqslant \delta,$$
$$|a_2x + b_2y - c_2| \leqslant \delta,$$

with $a_1b_2 \neq a_2b_1$. Sketch this set and show that it is possible to find $(x_1, y_1), (x_2, y_2) \in A$ with

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 \geqslant \frac{8\delta^2}{|a_1b_2 - a_2b_1|}.$$

9 Let $(G, *)$ and $(H, \circ)$ be two groups and $G \times H$ the set of ordered pairs (g, h) with $g \in G$ and $h \in H$. A multiplication on $G \times H$ is defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 * g_2, h_1 \circ h_2)$$

for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Show that, with this multiplication, $G \times H$ is a group.

State whether the following are true or false and prove your answers.

(i) $G \times H$ is abelian if and only if both G and H are abelian.

(ii) $G \times H$ contains a subgroup isomorphic to G.

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_4$.

(iv) $S_2 \times S_3$ is isomorphic to S_6.

[$\mathbb{Z}_n$ is the cyclic group of order n, and S_n is the permutation group on n objects.]

10 The Bernoulli polynomials $P_n(x)$, where n is a non-negative integer, are defined by $P_0(x) = 1$ and, for $n \geqslant 1$,

$$\frac{\mathrm{d}P_n}{\mathrm{d}x} = n\,P_{n-1}(x), \qquad \int_0^1 P_n(x)\,\mathrm{d}x = 0.$$

Show, by using induction or otherwise, that

$$P_n(x+1) - P_n(x) = n\,x^{n-1}, \qquad \text{for } n \geqslant 1.$$

Deduce that

$$n\sum_{m=0}^{k} m^{n-1} = P_n(k+1) - P_n(0).$$

Hence show that $\displaystyle\sum_{m=0}^{1000} m^3 = (500\,500)^2$.

11 A woman stands in a field at a distance of a m from the straight bank of a river which flows with negligible speed. She sees her frightened child clinging to a tree stump standing in the river b m downstream from where she stands and c m from the bank. She runs at a speed of u m s^{-1} and swims at v m s^{-1} in straight lines. Find an equation to be satisfied by x, where x m is the distance upstream from the stump at which she should enter the river if she is to reach the child in the shortest possible time.

Suppose now that the river flows with speed v m s^{-1} and the stump remains fixed. Show that, in this case, x must satisfy the equation

$$2vx^2(b-x) = u(x^2 - c^2)[a^2 + (b-x)^2]^{\frac{1}{2}}.$$

For this second case, draw sketches of the woman's path for the three possibilities $b > c$, $b = c$ and $b < c$.

12 A firework consists of a uniform rod of mass M and length $2a$, pivoted smoothly at one end so that it can rotate in a fixed horizontal plane, and a rocket attached to the other end. The rocket is a uniform rod of mass $m(t)$ and length $2l(t)$, with $m(t) = 2\alpha l(t)$ and α constant. It is attached to the rod by its front end and it lies at right angles to the rod in the rod's plane of rotation. The rocket burns fuel in such a way that $\mathrm{d}m/\mathrm{d}t = -\alpha\beta$, with β constant. The burnt fuel is ejected from the back of the rocket, with speed u and directly backwards relative to the rocket. Show that, until the fuel is exhausted, the firework's angular velocity ω at time t satisfies

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3\alpha\beta a u}{2[Ma^2 + 2\alpha l\{3a^2 + l^2\}]}.$$

13 A uniform rod, of mass $3m$ and length $2a$, is freely hinged at one end and held by the other end in a horizontal position. A rough particle, of mass m, is placed on the rod at its mid-point. If the free end is then released, prove that, until the particle begins to slide on the rod, the inclination θ of the rod to the horizontal satisfies the equation

$$5a\dot{\theta}^2 = 8g\sin\theta.$$

The coefficient of friction between the particle and the rod is $\frac{1}{2}$. Show that, when the particle begins to slide, $\tan\theta = \frac{1}{26}$.

14 It is given that the gravitational force between a disc, of radius a, thickness δx and uniform density ρ, and a particle of mass m at a distance $b\,(\geqslant 0)$ from the disc on its axis is

$$2\pi mk\rho\delta x\left[1 - \frac{b}{(a^2+b^2)^{\frac{1}{2}}}\right],$$

where k is a constant. Show that the gravitational force on a particle of mass m at the surface of a uniform sphere of mass M and radius r is kmM/r^2. Deduce that in a spherical cloud of particles of uniform density, which all attract one another gravitationally, the radius r and inward velocity $v\ (= -\mathrm{d}r/\mathrm{d}t)$ of a particle at the surface satisfy the equation

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{kM}{r^2},$$

where M is the mass of the cloud.

At time $t = 0$, the cloud is instantaneously at rest and has radius R. Show that $r = R\cos^2\alpha$ after a time

$$\left[\frac{R^3}{2kM}\right]^{\frac{1}{2}}(\alpha + \tfrac{1}{2}\sin 2\alpha).$$

15 A patient arrives with blue thumbs at the doctor's surgery. With probability p the patient is suffering from Fenland fever and requires treatment costing £100. With probability $1-p$ he is suffering only from Steppe syndrome and will get better anyway. A test exists which infallibly gives positive results if the patient is suffering from Fenland fever but also has probability q of giving positive results if the patient is not. The test costs £10. The doctor decides to proceed as follows. She will give the test repeatedly until *either* the last test is negative, in which case she dismisses the patient with kind words, *or* she has given the test n times with positive results each time, in which case she gives the treatment. In the case $n = 0$, she treats the patient at once. She wishes to minimise the expected cost £E_n to the National Health Service.

(i) Show that

$$E_{n+1} - E_n = 10p - 10(1-p)q^n(9-10q),$$

and deduce that if $p = 10^{-4}$, $q = 10^{-2}$, she should choose $n = 3$.

(ii) Show that if q is larger than some fixed value q_0, to be determined explicitly, then, whatever the value of p, she should choose $n = 0$.

16 (*a*) $X_1, X_2, \ldots, X_n$ are independent identically distributed random variables drawn from a uniform distribution on [0, 1]. The random variables A and B are defined by

$$A = \min(X_1, \ldots, X_n), \qquad B = \max(X_1, \ldots, X_n).$$

For any fixed k, such that $0 < k < \frac{1}{2}$, let

$$p_n = \mathrm{P}(A \leqslant k \text{ and } B \geqslant 1 - k).$$

What happens to p_n as $n \to \infty$? Comment briefly on this result.

(*b*) Lord Copper, the celebrated and imperious newspaper proprietor, has decided to run a lottery in which each of the 4 000 000 readers of his newspaper will have an equal probability p of winning £1 000 000 and their chances of winning will be independent. He has fixed all the details leaving to you, his subordinate, only the task of choosing p. If nobody wins £1 000 000, you will be sacked, and if more than two readers win £1 000 000, you will also be sacked. Explaining your reasoning, show that however you choose p, you will have less than a 60% chance of keeping your job.