

**Sixth Term Examination Papers**  
**MATHEMATICS 3**  
**MONDAY 21 JUNE 2010**

**9475**  
Afternoon  
Time: 3 hours

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Additional Materials: Answer Paper  
Formulae Booklet

Candidates may **not** use a calculator

**INSTRUCTIONS TO CANDIDATES**

**Please read this page carefully, but do not open this question paper until you are told that you may do so.**

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

**INFORMATION FOR CANDIDATES**

Each question is marked out of 20. There is no restriction of choice.

You will be assessed on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

**Calculators are not permitted.**

**Please wait to be told you may begin before turning this page.**

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**This question paper consists of 8 printed pages and 4 blank pages.**

**[Turn over**

## Section A: Pure Mathematics

- 1** Let  $x_1, x_2, \dots, x_n$  and  $x_{n+1}$  be any fixed real numbers. The numbers  $A$  and  $B$  are defined by

$$A = \frac{1}{n} \sum_{k=1}^n x_k, \quad B = \frac{1}{n} \sum_{k=1}^n (x_k - A)^2,$$

and the numbers  $C$  and  $D$  are defined by

$$C = \frac{1}{n+1} \sum_{k=1}^{n+1} x_k, \quad D = \frac{1}{n+1} \sum_{k=1}^{n+1} (x_k - C)^2.$$

- (i) Express  $C$  in terms of  $A$ ,  $x_{n+1}$  and  $n$ .
- (ii) Show that  $B = \frac{1}{n} \sum_{k=1}^n x_k^2 - A^2$ .
- (iii) Express  $D$  in terms of  $B$ ,  $A$ ,  $x_{n+1}$  and  $n$ .

Hence show that  $(n+1)D \geq nB$  for all values of  $x_{n+1}$ , but that  $D < B$  if and only if

$$A - \sqrt{\frac{(n+1)B}{n}} < x_{n+1} < A + \sqrt{\frac{(n+1)B}{n}}.$$

- 2** In this question,  $a$  is a positive constant.

- (i) Express  $\cosh a$  in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x \cosh a + 1} dx = \frac{a}{2 \sinh a}.$$

- (ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x \sinh a - 1} dx$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2 \cosh a + 1} dx.$$

- 3** For any given positive integer  $n$ , a number  $a$  (which may be complex) is said to be a *primitive  $n$ th root of unity* if  $a^n = 1$  and there is no integer  $m$  such that  $0 < m < n$  and  $a^m = 1$ . Write down the two primitive 4th roots of unity.

Let  $C_n(x)$  be the polynomial such that the roots of the equation  $C_n(x) = 0$  are the primitive  $n$ th roots of unity, the coefficient of the highest power of  $x$  is one and the equation has no repeated roots. Show that  $C_4(x) = x^2 + 1$ .

- (i) Find  $C_1(x)$ ,  $C_2(x)$ ,  $C_3(x)$ ,  $C_5(x)$  and  $C_6(x)$ , giving your answers as unfactorised polynomials.
- (ii) Find the value of  $n$  for which  $C_n(x) = x^4 + 1$ .
- (iii) Given that  $p$  is prime, find an expression for  $C_p(x)$ , giving your answer as an unfactorised polynomial.
- (iv) Prove that there are no positive integers  $q$ ,  $r$  and  $s$  such that  $C_q(x) \equiv C_r(x)C_s(x)$ .
- 4** (i) The number  $\alpha$  is a common root of the equations  $x^2 + ax + b = 0$  and  $x^2 + cx + d = 0$  (that is,  $\alpha$  satisfies both equations). Given that  $a \neq c$ , show that

$$\alpha = -\frac{b-d}{a-c}.$$

Hence, or otherwise, show that the equations have at least one common root if and only if

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

Does this result still hold if the condition  $a \neq c$  is not imposed?

- (ii) Show that the equations  $x^2 + ax + b = 0$  and  $x^3 + (a+1)x^2 + qx + r = 0$  have at least one common root if and only if

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0.$$

Hence, or otherwise, find the values of  $b$  for which the equations  $2x^2 + 5x + 2b = 0$  and  $2x^3 + 7x^2 + 5x + 1 = 0$  have at least one common root.

- 5 The vertices  $A$ ,  $B$ ,  $C$  and  $D$  of a square have coordinates  $(0, 0)$ ,  $(a, 0)$ ,  $(a, a)$  and  $(0, a)$ , respectively. The points  $P$  and  $Q$  have coordinates  $(an, 0)$  and  $(0, am)$  respectively, where  $0 < m < n < 1$ . The line  $CP$  produced meets  $DA$  produced at  $R$  and the line  $CQ$  produced meets  $BA$  produced at  $S$ . The line  $PQ$  produced meets the line  $RS$  produced at  $T$ . Show that  $TA$  is perpendicular to  $AC$ .

Explain how, given a square of area  $a^2$ , a square of area  $2a^2$  may be constructed using only a straight-edge.

[**Note:** a straight-edge is a ruler with no markings on it; no measurements (and no use of compasses) are allowed in the construction.]

- 6 The points  $P$ ,  $Q$  and  $R$  lie on a sphere of unit radius centred at the origin,  $O$ , which is fixed. Initially,  $P$  is at  $P_0(1, 0, 0)$ ,  $Q$  is at  $Q_0(0, 1, 0)$  and  $R$  is at  $R_0(0, 0, 1)$ .
- (i) The sphere is then rotated about the  $z$ -axis, so that the line  $OP$  turns directly towards the positive  $y$ -axis through an angle  $\phi$ . The position of  $P$  after this rotation is denoted by  $P_1$ . Write down the coordinates of  $P_1$ .
- (ii) The sphere is now rotated about the line in the  $x$ - $y$  plane perpendicular to  $OP_1$ , so that the line  $OP$  turns directly towards the positive  $z$ -axis through an angle  $\lambda$ . The position of  $P$  after this rotation is denoted by  $P_2$ . Find the coordinates of  $P_2$ . Find also the coordinates of the points  $Q_2$  and  $R_2$ , which are the positions of  $Q$  and  $R$  after the two rotations.
- (iii) The sphere is now rotated for a third time, so that  $P$  returns from  $P_2$  to its original position  $P_0$ . During the rotation,  $P$  remains in the plane containing  $P_0$ ,  $P_2$  and  $O$ . Show that the angle of this rotation,  $\theta$ , satisfies

$$\cos \theta = \cos \phi \cos \lambda,$$

and find a vector in the direction of the axis about which this rotation takes place.

- 7 Given that  $y = \cos(m \arcsin x)$ , for  $|x| < 1$ , prove that

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0.$$

Obtain a similar equation relating  $\frac{d^3 y}{dx^3}$ ,  $\frac{d^2 y}{dx^2}$  and  $\frac{dy}{dx}$ , and a similar equation relating  $\frac{d^4 y}{dx^4}$ ,  $\frac{d^3 y}{dx^3}$  and  $\frac{d^2 y}{dx^2}$ .

Conjecture and prove a relation between  $\frac{d^{n+2} y}{dx^{n+2}}$ ,  $\frac{d^{n+1} y}{dx^{n+1}}$  and  $\frac{d^n y}{dx^n}$ .

Obtain the first three non-zero terms of the Maclaurin series for  $y$ . Show that, if  $m$  is an even integer,  $\cos m\theta$  may be written as a polynomial in  $\sin \theta$  beginning

$$1 - \frac{m^2 \sin^2 \theta}{2!} + \frac{m^2(m^2 - 2^2) \sin^4 \theta}{4!} - \dots \quad (|\theta| < \frac{1}{2}\pi)$$

State the degree of the polynomial.

- 8 Given that  $P(x) = Q(x)R'(x) - Q'(x)R(x)$ , write down an expression for

$$\int \frac{P(x)}{(Q(x))^2} dx.$$

- (i) By choosing the function  $R(x)$  to be of the form  $a + bx + cx^2$ , find

$$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2} dx.$$

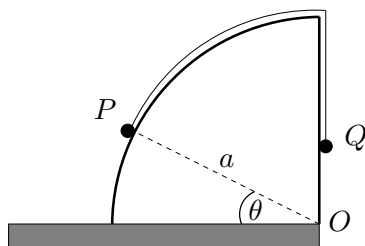
Show that the choice of  $R(x)$  is not unique and, by comparing the two functions  $R(x)$  corresponding to two different values of  $a$ , explain how the different choices are related.

- (ii) Find the general solution of

$$(1 + \cos x + 2 \sin x) \frac{dy}{dx} + (\sin x - 2 \cos x)y = 5 - 3 \cos x + 4 \sin x.$$

## Section B: Mechanics

9



The diagram shows two particles,  $P$  and  $Q$ , connected by a light inextensible string which passes over a smooth block fixed to a horizontal table. The cross-section of the block is a quarter circle with centre  $O$ , which is at the edge of the table, and radius  $a$ . The angle between  $OP$  and the table is  $\theta$ . The masses of  $P$  and  $Q$  are  $m$  and  $M$ , respectively, where  $m < M$ .

Initially,  $P$  is held at rest on the table and in contact with the block,  $Q$  is vertically above  $O$ , and the string is taut. Then  $P$  is released. Given that, in the subsequent motion,  $P$  remains in contact with the block as  $\theta$  increases from 0 to  $\frac{1}{2}\pi$ , find an expression, in terms of  $m$ ,  $M$ ,  $\theta$  and  $g$ , for the normal reaction of the block on  $P$  and show that

$$\frac{m}{M} \geq \frac{\pi - 1}{3}.$$

- 10** A small bead  $B$ , of mass  $m$ , slides without friction on a fixed horizontal ring of radius  $a$ . The centre of the ring is at  $O$ . The bead is attached by a light elastic string to a fixed point  $P$  in the plane of the ring such that  $OP = b$ , where  $b > a$ . The natural length of the elastic string is  $c$ , where  $c < b - a$ , and its modulus of elasticity is  $\lambda$ . Show that the equation of motion of the bead is

$$ma\ddot{\phi} = -\lambda \left( \frac{a \sin \phi}{c \sin \theta} - 1 \right) \sin(\theta + \phi),$$

where  $\theta = \angle BPO$  and  $\phi = \angle BOP$ .

Given that  $\theta$  and  $\phi$  are small, show that  $a(\theta + \phi) \approx b\theta$ . Hence find the period of small oscillations about the equilibrium position  $\theta = \phi = 0$ .

- 11** A bullet of mass  $m$  is fired horizontally with speed  $u$  into a wooden block of mass  $M$  at rest on a horizontal surface. The coefficient of friction between the block and the surface is  $\mu$ . While the bullet is moving through the block, it experiences a constant force of resistance to its motion of magnitude  $R$ , where  $R > (M + m)\mu g$ . The bullet moves horizontally in the block and does not emerge from the other side of the block.

- (i) Show that the magnitude,  $a$ , of the deceleration of the bullet relative to the block while the bullet is moving through the block is given by

$$a = \frac{R}{m} + \frac{R - (M + m)\mu g}{M}.$$

- (ii) Show that the common speed,  $v$ , of the block and bullet when the bullet stops moving through the block satisfies

$$av = \frac{Ru - (M + m)\mu gu}{M}.$$

- (iii) Obtain an expression, in terms of  $u$ ,  $v$  and  $a$ , for the distance moved by the block while the bullet is moving through the block.

- (iv) Show that the total distance moved by the block is

$$\frac{mu v}{2(M + m)\mu g}.$$

Describe briefly what happens if  $R < (M + m)\mu g$ .

## Section C: Probability and Statistics

- 12** The infinite series  $S$  is given by

$$S = 1 + (1 + d)r + (1 + 2d)r^2 + \cdots + (1 + nd)r^n + \cdots ,$$

for  $|r| < 1$ . By considering  $S - rS$ , or otherwise, prove that

$$S = \frac{1}{1 - r} + \frac{rd}{(1 - r)^2} .$$

Arthur and Boadicea shoot arrows at a target. The probability that an arrow shot by Arthur hits the target is  $a$ ; the probability that an arrow shot by Boadicea hits the target is  $b$ . Each shot is independent of all others. Prove that the expected number of shots it takes Arthur to hit the target is  $1/a$ .

Arthur and Boadicea now have a contest. They take alternate shots, with Arthur going first. The winner is the one who hits the target first. The probability that Arthur wins the contest is  $\alpha$  and the probability that Boadicea wins is  $\beta$ . Show that

$$\alpha = \frac{a}{1 - a'b'} ,$$

where  $a' = 1 - a$  and  $b' = 1 - b$ , and find  $\beta$ .

Show that the expected number of shots in the contest is  $\frac{\alpha}{a} + \frac{\beta}{b}$ .

- 13** In this question,  $\text{Corr}(U, V)$  denotes the product moment correlation coefficient between the random variables  $U$  and  $V$ , defined by

$$\text{Corr}(U, V) \equiv \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U)\text{Var}(V)}} .$$

The independent random variables  $Z_1$ ,  $Z_2$  and  $Z_3$  each have expectation 0 and variance 1. What is the value of  $\text{Corr}(Z_1, Z_2)$ ?

Let  $Y_1 = Z_1$  and let

$$Y_2 = \rho_{12}Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}}Z_2 ,$$

where  $\rho_{12}$  is a given constant with  $-1 < \rho_{12} < 1$ . Find  $E(Y_2)$ ,  $\text{Var}(Y_2)$  and  $\text{Corr}(Y_1, Y_2)$ .

Now let  $Y_3 = aZ_1 + bZ_2 + cZ_3$ , where  $a$ ,  $b$  and  $c$  are real constants and  $c \geq 0$ . Given that  $E(Y_3) = 0$ ,  $\text{Var}(Y_3) = 1$ ,  $\text{Corr}(Y_1, Y_3) = \rho_{13}$  and  $\text{Corr}(Y_2, Y_3) = \rho_{23}$ , express  $a$ ,  $b$  and  $c$  in terms of  $\rho_{23}$ ,  $\rho_{13}$  and  $\rho_{12}$ .

Given constants  $\mu_i$  and  $\sigma_i$ , for  $i = 1, 2$  and  $3$ , give expressions in terms of the  $Y_i$  for random variables  $X_i$  such that  $E(X_i) = \mu_i$ ,  $\text{Var}(X_i) = \sigma_i^2$  and  $\text{Corr}(X_i, X_j) = \rho_{ij}$ .



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