

**SIXTH TERM EXAMINATION PAPERS**  
administered by the Oxford and Cambridge Schools Examination Board  
on behalf of the Cambridge Colleges

9475

**MATHEMATICS III**

Friday 28 June 1996, afternoon

3 hours

*Additional materials:*

*script paper; graph paper; MF(STEP)1.*

*To be brought by candidate: electronic calculator;  
standard geometrical instruments.*

*All questions carry equal weight.*

*You are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.*

*Candidates may attempt as many questions as they wish but marks will be assessed on the six questions best answered.*

*You are provided with Mathematical Formulae and Tables MF(STEP)1.*

*Electronic calculators may be used.*

**Section A: Pure Mathematics**

- 1 Define  $\cosh x$  and  $\sinh x$  in terms of exponentials and prove, from your definitions, that

$$\cosh^4 x - \sinh^4 x = \cosh 2x$$

and

$$\cosh^4 x + \sinh^4 x = \frac{1}{4} \cosh 4x + \frac{3}{4}.$$

Find  $a_0, a_1, \dots, a_n$  in terms of  $n$  such that

$$\cosh^n x = a_0 + a_1 \cosh x + a_2 \cosh 2x + \dots + a_n \cosh nx.$$

Hence, or otherwise, find expressions for  $\cosh^{2m} x - \sinh^{2m} x$  and  $\cosh^{2m} x + \sinh^{2m} x$ , in terms of  $\cosh kx$ , where  $k = 0, \dots, 2m$ .

- 2 For all values of  $a$  and  $b$ , either solve the simultaneous equations

$$x + y + az = 2$$

$$x + ay + z = 2$$

$$2x + y + z = 2b$$

or prove that they have no solution.

- 3 Find

$$\int_0^\theta \frac{1}{1 - a \cos x} dx,$$

where  $0 < \theta < \pi$  and  $-1 < a < 1$ .

Hence show that

$$\int_0^{\pi/2} \frac{1}{2 - a \cos x} dx = \frac{2}{\sqrt{4 - a^2}} \tan^{-1} \sqrt{\frac{2 + a}{2 - a}},$$

and also that

$$\int_0^{3\pi/4} \frac{1}{\sqrt{2} + \cos x} dx = \frac{\pi}{2}.$$

4 Find the integers  $k$  satisfying the inequality  $k \leq 2(k - 2)$ .

Given that  $N$  is a strictly positive integer consider the problem of finding strictly positive integers whose sum is  $N$  and whose product is as large as possible. Call this largest possible product  $P(N)$ . Show that  $P(5) = 2 \times 3$ ,  $P(6) = 3^2$ ,  $P(7) = 2^2 \times 3$ ,  $P(8) = 2 \times 3^2$  and  $P(9) = 3^3$ .

Find  $P(1000)$  explaining your reasoning carefully.

5 Show, using de Moivre's Theorem, or otherwise, that

$$\tan 7\theta = \frac{t(t^6 - 21t^4 + 35t^2 - 7)}{7t^6 - 35t^4 + 21t^2 - 1},$$

where  $t = \tan \theta$ .

(i) By considering the equation  $\tan 7\theta = 0$ , or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{7}\right), \tan^2\left(\frac{2\pi}{7}\right) \text{ and } \tan^2\left(\frac{3\pi}{7}\right)$$

and deduce the value of

$$\tan\left(\frac{\pi}{7}\right) \tan\left(\frac{2\pi}{7}\right) \tan\left(\frac{3\pi}{7}\right).$$

(ii) Find, without using a calculator, the value of

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right).$$

6 (i) Let  $S$  be the set of matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where  $a$  is any real non-zero number. Show that  $S$  is closed under matrix multiplication and, further, that  $S$  is a group under matrix multiplication.

(ii) Let  $G$  be a set of  $n \times n$  matrices which is a group under matrix multiplication, with identity element  $\mathbf{E}$ . By considering equations of the form  $\mathbf{BC} = \mathbf{D}$  for suitable elements  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{D}$  of  $G$ , show that if a given element  $\mathbf{A}$  of  $G$  is a singular matrix (i.e.  $\det \mathbf{A} = 0$ ), then all elements of  $G$  are singular. Give, with justification, an example of such a group of singular matrices in the case  $n = 3$ .

7 (i) If  $x + y + z = \alpha$ ,  $xy + yz + zx = \beta$  and  $xyz = \gamma$ , find numbers  $A$ ,  $B$  and  $C$  such that

$$x^3 + y^3 + z^3 = A\alpha^3 + B\alpha\beta + C\gamma.$$

Solve the equations

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 3 \\ x^3 + y^3 + z^3 &= 4. \end{aligned}$$

(ii) The area of a triangle whose sides are  $a$ ,  $b$  and  $c$  is given by the formula

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where  $s$  is the semi-perimeter  $\frac{1}{2}(a+b+c)$ . If  $a$ ,  $b$  and  $c$  are the roots of the equation

$$x^3 - 16x^2 + 81x - 128 = 0,$$

find the area of the triangle.

8 A transformation  $T$  of the real numbers is defined by

$$y = T(x) = \frac{ax - b}{cx - d},$$

where  $a, b, c, d$  are real numbers such that  $ad \neq bc$ . Find all numbers  $x$  such that  $T(x) = x$ . Show that the inverse operation,  $x = T^{-1}(y)$  expressing  $x$  in terms of  $y$  is of the same form as  $T$  and find corresponding numbers  $a', b', c', d'$ .

Let  $S_r$  denote the set of all real numbers excluding  $r$ . Show that, if  $c \neq 0$ , there is a value of  $r$  such that  $T$  is defined for all  $x \in S_r$  and find the image  $T(S_r)$ . What is the corresponding result if  $c = 0$ ?

If  $T_1$ , given by numbers  $a_1, b_1, c_1, d_1$ , and  $T_2$ , given by numbers  $a_2, b_2, c_2, d_2$ , are two such transformations, show that their composition  $T_3$ , defined by  $T_3(x) = T_2(T_1(x))$ , is of the same form.

Find necessary and sufficient conditions on the numbers  $a, b, c, d$  for  $T^2$ , the composition of  $T$  with itself, to be the identity. Hence, or otherwise, find transformations  $T_1, T_2$  and their composition  $T_3$  such that  $T_1^2$  and  $T_2^2$  are each the identity but  $T_3^2$  is not.

## Section B: Mechanics

**9** A particle of mass  $m$  is at rest on top of a smooth fixed sphere of radius  $a$ . Show that, if the particle is given a small displacement, it reaches the horizontal plane through the centre of the sphere at a distance

$$a(5\sqrt{5} + 4\sqrt{23})/27$$

from the centre of the sphere.

[Air resistance should be neglected.]

**10** Two rough solid circular cylinders, of equal radius and length and of uniform density, lie side by side on a rough plane inclined at an angle  $\alpha$  to the horizontal, where  $0 < \alpha < \pi/2$ . Their axes are horizontal and they touch along their entire length. The weight of the upper cylinder is  $W_1$  and the coefficient of friction between it and the plane is  $\mu_1$ . The corresponding quantities for the lower cylinder are  $W_2$  and  $\mu_2$  respectively and the coefficient of friction between the two cylinders is  $\mu$ . Show that for equilibrium to be possible:

- (i)  $W_1 > W_2$  ;
- (ii)  $\mu \geq \frac{W_1 + W_2}{W_1 - W_2}$  ;
- (iii)  $\mu_1 \geq \left( \frac{2W_1 \cot \alpha}{W_1 + W_2} - 1 \right)^{-1}$  .

Find the similar inequality to (iii) for  $\mu_2$ .

**11** A smooth circular wire of radius  $a$  is held fixed in a vertical plane with light elastic strings of natural length  $a$  and modulus  $\lambda$  attached to the upper and lower extremities,  $A$  and  $C$  respectively, of the vertical diameter. The other ends of the two strings are attached to a small ring  $B$  of mass  $m$  which is free to slide on the wire. Show that, while both strings remain taut, the equation for the motion of the ring is

$$2ma\ddot{\theta} = \lambda(\cos \theta - \sin \theta) - mg \sin 2\theta,$$

where  $\theta$  is the angle  $\widehat{CAB}$ .

Initially the system is at rest in equilibrium with  $\sin \theta = 3/5$ . Deduce that  $5\lambda = 24mg$ .

The ring is now displaced slightly. Show that in the ensuing motion it will oscillate with period approximately

$$10\pi\sqrt{\frac{a}{91g}}.$$

## Section C: Probability &amp; Statistics

**12** It has been observed that Professor Ecks proves three types of theorems: 1, those that are correct and new; 2, those that are correct, but already known; 3, those that are false. It has also been observed that, if a certain of her theorems is of type  $i$ , then her next theorem is of type  $j$  with probability  $p_{ij}$ , where  $p_{ij}$  is the entry in the  $i$ th row and  $j$ th column of the following array:

$$\begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

Let  $a_i$ ,  $i = 1, 2, 3$ , be the probability that a given theorem is of type  $i$ , and let  $b_j$  be the consequent probability that the next theorem is of type  $j$ .

- (i) Explain why  $b_j = a_1p_{1j} + a_2p_{2j} + a_3p_{3j}$ .
- (ii) Find values of  $a_1$ ,  $a_2$  and  $a_3$  such that  $b_i = a_i$  for  $i = 1, 2, 3$ .
- (iii) For these values of the  $a_i$  find the probabilities  $q_{ij}$  that, if a particular theorem is of type  $j$ , then the *preceding* theorem was of type  $i$ .

**13** Let  $X$  be a random variable which takes only the finite number of different possible real values  $x_1, x_2, \dots, x_n$ . Define the expectation  $E(X)$  and the variance  $\text{var}(X)$  of  $X$ . Show that, if  $a$  and  $b$  are real numbers, then  $E(aX + b) = aE(X) + b$  and express  $\text{var}(aX + b)$  similarly in terms of  $\text{var}(X)$ .

Let  $\lambda$  be a positive real number. By considering the contribution to  $\text{var}(X)$  of those  $x_i$  for which  $|x_i - E(X)| \geq \lambda$ , or otherwise, show that

$$P[|X - E(X)| \geq \lambda] \leq \frac{\text{var}(X)}{\lambda^2}.$$

Let  $k$  be a real number satisfying  $k \geq \lambda$ . If  $|x_i - E(X)| \leq k$  for all  $i$ , show that

$$P[|X - E(X)| \geq \lambda] \geq \frac{\text{var}(X) - \lambda^2}{k^2 - \lambda^2}.$$

14 Whenever I go cycling I start with my bike in good working order. However if all is well at time  $t$ , the probability that I get a puncture in the small interval  $(t, t + \delta t)$  is  $\alpha \delta t$ . How many punctures can I expect to get on a journey during which my total cycling time is  $T$ ?

When I get a puncture I stop immediately to repair it and the probability that, if I am repairing it at time  $t$ , the repair will be completed in time  $(t, t + \delta t)$  is  $\beta \delta t$ . If  $p(t)$  is the probability that I am repairing a puncture at time  $t$ , write down an equation relating  $p(t)$  to  $p(t + \delta t)$ , and derive from this a differential equation relating  $p'(t)$  and  $p(t)$ . Show that

$$p(t) = \frac{\alpha}{\alpha + \beta} (1 - \exp(-(\alpha + \beta)t))$$

satisfies this differential equation with the appropriate initial condition.

Find an expression, involving  $\alpha$ ,  $\beta$  and  $T$ , for the time expected to be spent mending punctures during a journey of total time  $T$ . Hence, or otherwise, show that, the fraction of the journey expected to be spent mending punctures is given approximately by

$$\frac{\alpha T}{2} \quad \text{if } (\alpha + \beta)T \text{ is small,}$$

and by

$$\frac{\alpha}{\alpha + \beta} \quad \text{if } (\alpha + \beta)T \text{ is large.}$$