GCSE Maths – Geometry and Measures

Geometric Arguments and Proof (Higher)

Worksheet

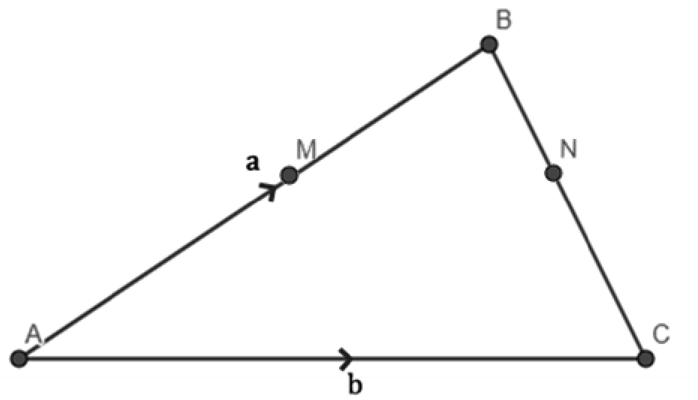
WORKED SOLUTIONS

This worksheet will show you how to work out different types of geometric arguments and proof questions. Each section contains a worked example, a question with hints and then questions for you to work through on your own.

Section A

Worked Example

In triangle ABC, $\overrightarrow{\mathrm{AB}} = \mathrm{a}$ and $\overrightarrow{\mathrm{AC}} = \mathrm{b}$. M and N are the midpoints of $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$ respectively. Prove that $\overrightarrow{\mathrm{MN}}$ is <u>not</u> parallel to $\overrightarrow{\mathrm{AB}}$.



Step 1: Find $\overrightarrow{\mathrm{AM}}$ and $\overrightarrow{\mathrm{CN}}$ in terms of **a** and **b**.

$$\overrightarrow{CB} = -\overrightarrow{AB} + \overrightarrow{AC} = -\boldsymbol{a} + \boldsymbol{b}$$

M is halfway along $\overrightarrow{AB}$ so:

$$\overrightarrow{AM} = \frac{1}{2}\left(\overrightarrow{AB}\right) = \frac{1}{2}\boldsymbol{a}$$

N is halfway along $\overrightarrow{CB}$ so:

$$\overrightarrow{CN} = \frac{1}{2}\left(\overrightarrow{CB}\right) = \frac{1}{2}\boldsymbol{b}$$

Step 2: Use these vectors to find the vector $\overrightarrow{\mathrm{MN}}$.

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AC} + \overrightarrow{CN} = -\overrightarrow{AM} + \overrightarrow{AC} + \overrightarrow{CN} = -\left(\frac{1}{2}\boldsymbol{a}\right) + \boldsymbol{b} + \frac{1}{2}\boldsymbol{b} = -\frac{1}{2}\boldsymbol{a} + \frac{3}{2}\boldsymbol{b}$$

Step 3: Show that $\overrightarrow{MN}$ is not a multiple of $\overrightarrow{\mathrm{AC}}$ and conclude what this means.

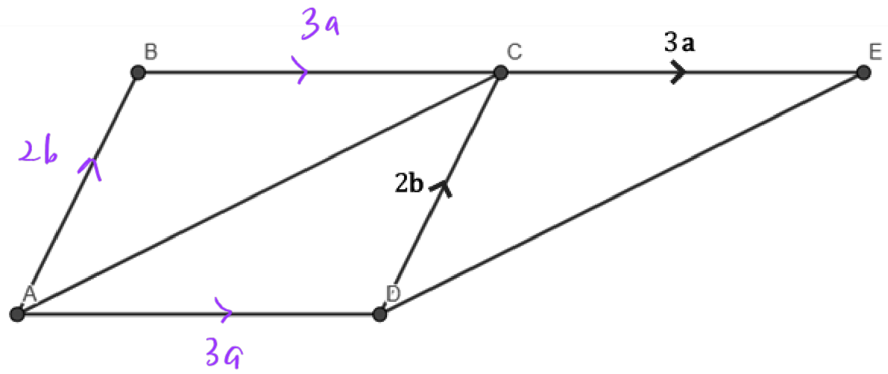
$$\left(-\tfrac{1}{2}\boldsymbol{a} + \tfrac{3}{2}\boldsymbol{b}\right) \div \boldsymbol{b} \neq a\ constant$$

We have shown that $\overrightarrow{MN}$ is not a constant multiple of $\overrightarrow{AC}$ and therefore $\overrightarrow{MN}$ and $\overrightarrow{AC}$ are not parallel.

Guided Example

ABCD is a parallelogram, and $\overrightarrow{\mathbf{BC}} = \overrightarrow{\mathbf{CE}}$.
$\overrightarrow{\mathbf{CE}} = 3\mathbf{a}$ and $\overrightarrow{\mathbf{DC}} = 2\mathbf{b}$.

Prove that $\overrightarrow{\mathbf{AC}}$ and $\overrightarrow{\mathbf{DE}}$ are parallel lines.



Step 1: Find the vectors $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of **a** and **b**.

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= 2b + 3a$
$\overrightarrow{AC} = 3a + 2b$

$\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD}$
$= (3a + 2b) + (-\overrightarrow{DC})$
$= (3a + 2b) - 2b$
$\overrightarrow{AD} = 3a$

Step 2: Use these to find the vector $\overrightarrow{DE}$.

$\overrightarrow{DE} = \overrightarrow{DC} + \overrightarrow{CE}$
$= 2b + 3a$
$\overrightarrow{DE} = 3a + 2b$

Step 3: Show that $\overrightarrow{AC}$ is a multiple of $\overrightarrow{DE}$ and conclude what this means.

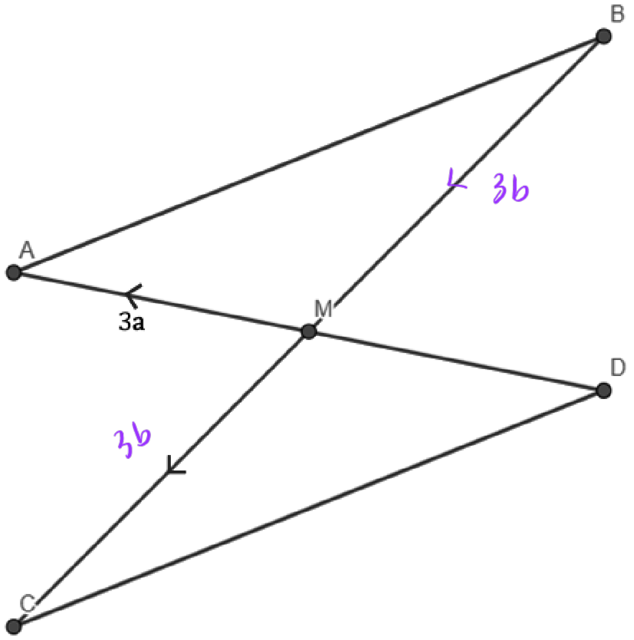
$\overrightarrow{AC} = 3a + 2b$
$\overrightarrow{DE} = 3a + 2b$
$\overrightarrow{AC} = \overrightarrow{DE}$

$\overrightarrow{AC}$ is equal to $\overrightarrow{DE}$ which means they share a common multiple of 1. This proves that $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel lines.

Now it's your turn!

If you get stuck, look back at the worked and guided examples.

1. Two triangles AMB and DMC are congruent. M is the midpoint of $\overrightarrow{\mathrm{BC}}$ and $\overrightarrow{\mathrm{DA}}$. $\overrightarrow{\mathrm{MA}} = 3\mathbf{a}$ and $\overrightarrow{\mathrm{MC}} = 3\mathbf{b}$. Prove that $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{CD}}$ are parallel.



$$\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MB}$$
$$= -3a + \overrightarrow{MB}$$
$$= -3a + (-3b)$$
$$= -3a - 3b$$

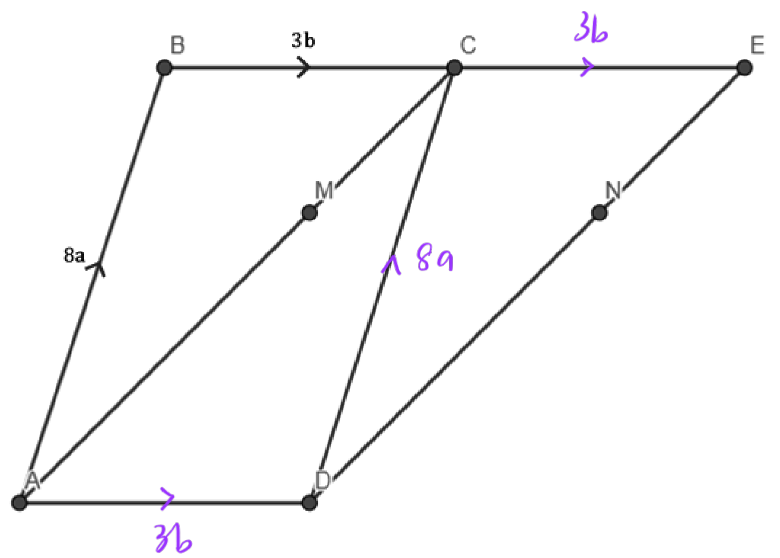
$\overrightarrow{MB} = \overrightarrow{CM}$ because M is the midpoint of $\overrightarrow{CB}$.

$$\overrightarrow{CD} = \overrightarrow{CM} + \overrightarrow{MD}$$
$$= -3b + \overrightarrow{MD}$$
$$= -3b + (-3a)$$
$$= -3b - 3a$$
$$= -3a - 3b$$

$\overrightarrow{MD} = \overrightarrow{AM}$ because M is the midpoint of $\overrightarrow{AD}$

$\overrightarrow{AB}$ is equal to $\overrightarrow{CD}$. This means that they share a common multiple of 1. This proves that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

2. ABCD is a parallelogram.
$\overrightarrow{AB} = 8\mathbf{a}$ and $\overrightarrow{BC} = 3\mathbf{b}$.
$\overrightarrow{CE}$ is an extension of $\overrightarrow{BC}$ such that $\overrightarrow{BC} = \overrightarrow{CE}$.
M splits $\overrightarrow{AC}$ such that $\overrightarrow{AM} : \overrightarrow{MC} = 3 : 1$ and N splits $\overrightarrow{DE}$ such that $\overrightarrow{DN} : \overrightarrow{NE} = 3 : 1$.
Prove that $\overrightarrow{AD}$ and $\overrightarrow{MN}$ are parallel.



$\overrightarrow{AD} = \overrightarrow{BC}$ because it is a parallelogram.
Hence, $\overrightarrow{AD} = 3b$

To find $\overrightarrow{MN}$, we need vectors of $\overrightarrow{MC}$, $\overrightarrow{CE}$ and $\overrightarrow{EN}$.

$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CE} + \overrightarrow{EN}$

$= \frac{1}{4}(\overrightarrow{AC}) + 3b + \frac{1}{4}(\overrightarrow{ED})$

$= \frac{1}{4}(\overrightarrow{AB} + \overrightarrow{BC}) + 3b + \frac{1}{4}(\overrightarrow{EC} + \overrightarrow{CD})$

$= \frac{1}{4}(8a + 3b) + 3b + \frac{1}{4}(-3b + (-8a))$

$= \frac{1}{4}(8a+3b) + 3b + \frac{1}{4}(-3b-8a)$

$= 2a + \frac{3}{4}b + 3b - \frac{3}{4}b - 2a$

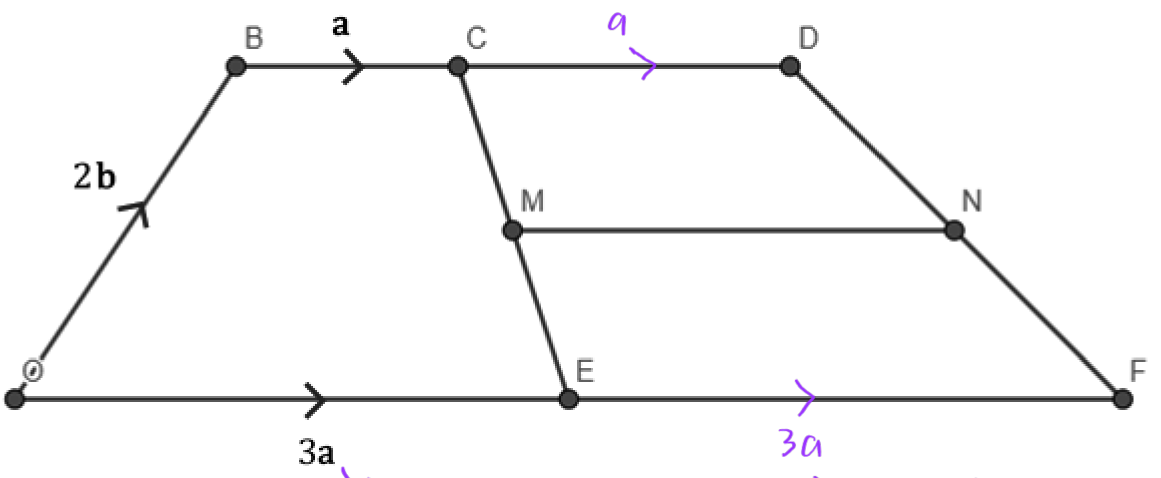
$\overrightarrow{MN} = 3b$

$\overrightarrow{MN} = 3b$
$\overrightarrow{AD} = 3b$

This means $\overrightarrow{MN} = \overrightarrow{AD}$.

We have proven that $\overrightarrow{MN}$ shares a common multiple of 1 with vector $\overrightarrow{AD}$. This means that $\overrightarrow{MN}$ is parallel to $\overrightarrow{AD}$.

3. OBDF is a trapezium.
$\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$ and $\overrightarrow{OE} = 3\mathbf{a}$.
C is the midpoint of $\overrightarrow{BD}$ and E is the midpoint of $\overrightarrow{OF}$.
M is the midpoint of $\overrightarrow{CE}$ and N is the midpoint of $\overrightarrow{DF}$.



when vectors are in opposite direction, the sign changes, (eg. $\overrightarrow{OE} = 3a$, $\overrightarrow{EO} = -3a$)

a) Find the vector $\overrightarrow{CE}$ in terms of **a** and **b**.

$$\overrightarrow{CE} = \overrightarrow{CB} + \overrightarrow{BO} + \overrightarrow{OE}$$
$$= (-a) + (-2b) + 3a$$
$$= -a - 2b + 3a$$
$$= 2a - 2b$$

b) Find the vector $\overrightarrow{DF}$ in terms of **a** and **b**.

$$\overrightarrow{DF} = \overrightarrow{DB} + \overrightarrow{BO} + \overrightarrow{OF}$$
$$= (-2a) + (-2b) + (6a)$$
$$= -2a - 2b + 6a$$
$$= 4a - 2b$$

c) Prove that $\overrightarrow{MN}$ is parallel to $\overrightarrow{OF}$.

$$\overrightarrow{OF} = 3a + 3a$$
$$= 6a$$

$$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CD} + \overrightarrow{DN}$$
$$= \frac{\overrightarrow{EC}}{2} + a + \frac{\overrightarrow{DF}}{2}$$
$$\overrightarrow{MN} = \frac{-(2a-2b)}{2} + a + \frac{4a-2b}{2}$$
$$= -a + b + a + 2a - b$$
$$= 2a$$

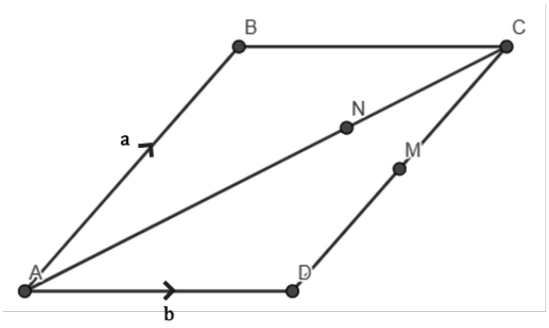
$$\overrightarrow{MN} = \frac{1}{3}(\overrightarrow{OF})$$

We have proven that $\overrightarrow{MN}$ is $\frac{1}{3}$ of the vector $\overrightarrow{OF}$. This proves that $\overrightarrow{MN}$ is parallel to $\overrightarrow{OF}$.

Section B

Worked Example

The shape ABCD is a parallelogram. $\overrightarrow{\mathbf{AB}}$ is represented by the vector $\mathbf{a}$ and $\overrightarrow{\mathbf{AD}}$ is represented by the vector $\mathbf{b}$. M is the midpoint of $\overrightarrow{\mathbf{BC}}$ and N is a point on $\overrightarrow{\mathbf{AC}}$ such that $\overrightarrow{\mathbf{AN}} : \overrightarrow{\mathbf{NC}} = 2 : 1$. Prove that the points B, N and M lie on a straight line.



Step 1: Find simplified expressions for the vectors $\overrightarrow{AN}$, $\overrightarrow{AM}$, $\overrightarrow{BN}$ and $\overrightarrow{BM}$. To prove that B, N and M are on a straight line, you must show through a series of logical steps that their vectors are multiples of each other.

$\overrightarrow{AB} = \overrightarrow{DC}$ *and* $\overrightarrow{AD} = \overrightarrow{BC}$ *because the shape is a parallelogram.*

Therefore, $\overrightarrow{DC} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{b}$.

N is $\frac{2}{3}$ *of the way along* $\overrightarrow{AC}$, *so:*

$$\overrightarrow{AN} = \frac{2}{3}\overrightarrow{AC}$$

$$\overrightarrow{AC} = \boldsymbol{a} + \boldsymbol{b}$$

$$\overrightarrow{AN} = \frac{2}{3}(\boldsymbol{a} + \boldsymbol{b}) = \frac{2}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$

M is halfway along $\overrightarrow{DC}$ *so:*

$$\overrightarrow{DM} = \frac{1}{2}\boldsymbol{a}$$

$$\overrightarrow{BN} = -\overrightarrow{AB} + \overrightarrow{AN} = -\boldsymbol{a} + \frac{2}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b} = -\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$$

$$\overrightarrow{BM} = -\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{DM} = -\boldsymbol{a} + \boldsymbol{b} + \frac{1}{2}\boldsymbol{a} = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b}$$

Step 2: To prove that the points lie on a straight line, show that $\overrightarrow{BM}$ *is a multiple of* $\overrightarrow{BN}$.

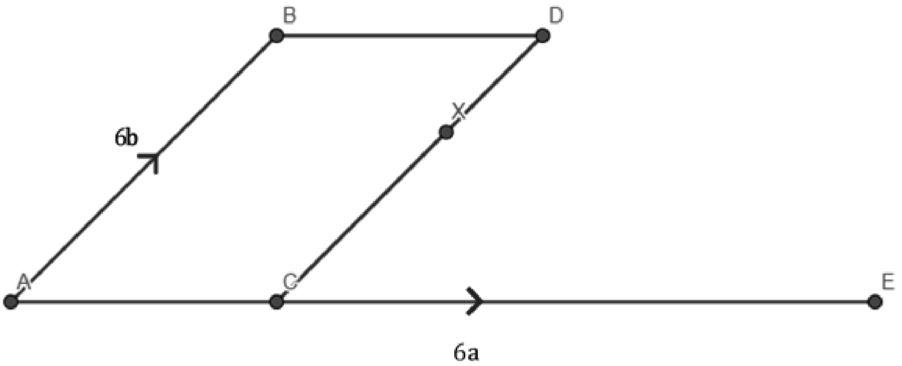
$$\frac{3}{2}\overrightarrow{BN} = \frac{3}{2}\left(-\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}\right) = -\frac{1}{2}\boldsymbol{a} + \boldsymbol{b} = \overrightarrow{BM}$$

Step 3: Conclude your proof.

Since $\overrightarrow{BM} = \frac{3}{2}\overrightarrow{BN}$, $\overrightarrow{BM}$ *and* $\overrightarrow{BN}$ *are parallel. They also share a common point B so the points B, N and M lie on a straight line.*

Guided Example

ABCD is a parallelogram.
$\overrightarrow{AB} = 6b$ and $\overrightarrow{AE} = 6a$
$\overrightarrow{CE}$ is an extension of the line $\overrightarrow{AC}$ such that $\overrightarrow{AC} : \overrightarrow{CE} = 1 : 2$.
The point X splits the line $\overrightarrow{CD}$ such that $\overrightarrow{CX} : \overrightarrow{XD} = 2 : 1$.
Prove that the points B, X and E lie on a straight line.



Step 1: Deduce $\overrightarrow{CD}$ and $\overrightarrow{BD}$ in terms of **a** and **b**.

$\overrightarrow{CD}$ is equal to $\overrightarrow{AB}$ since it is a parallelogram.

$\overrightarrow{CD} = 6b$

$\overrightarrow{BD} = \overrightarrow{AC}$

$= \frac{1}{3} \times 6a$

$\overrightarrow{BD} = 2a$

Step 2: Use the ratios given to find $\overrightarrow{DX}$ and $\overrightarrow{CE}$ in terms of **a** and **b**.

$\overrightarrow{DX} = \frac{1}{3} \times \overrightarrow{DC}$

$= \frac{1}{3} \times (-6b)$

$\overrightarrow{DX} = -2b$

$\overrightarrow{CE} = \frac{2}{3} \times \overrightarrow{AE}$

$= \frac{2}{3} \times 6a$

$\overrightarrow{CE} = 4a$

Step 3: Combine these vectors to find $\overrightarrow{BX}$ and $\overrightarrow{XE}$ (or $\overrightarrow{BE}$) in terms of **a** and **b**.

$\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AE}$

$= -6b + 6a$

$\overrightarrow{BX} = \overrightarrow{BD} + \overrightarrow{DX}$

$= 2a + (-2b)$

$= 2a - 2b$

$\overrightarrow{XE} = \overrightarrow{XC} + \overrightarrow{CE}$

$= -4b + 4a$

Step 4: Write a conclusion, showing that the vectors are multiples of each other and stating the common point. Explain what this means about the points.

$\overrightarrow{XE} = 2\overrightarrow{BX}$. Since $\overrightarrow{XE}$ is a multiple of $\overrightarrow{BX}$, this means that the lines are parallel. They also share a common point X, hence, the points E, B and X lies on a straight line.

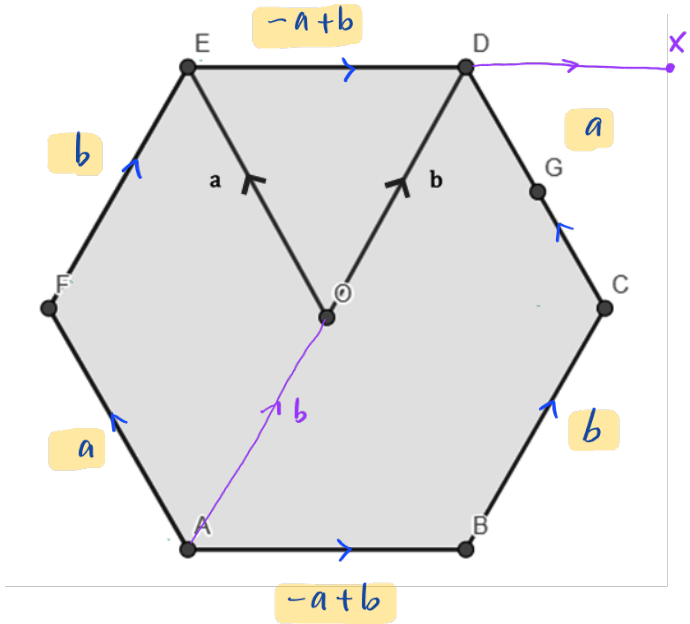
Now it's your turn!
If you get stuck, look back at the worked and guided examples.

4. ABCDEF is a regular hexagon with centre O.

$\overrightarrow{OE} = \boldsymbol{a}$ and $\overrightarrow{OD} = \boldsymbol{b}$

The point X lies on an extension of ED, such that $\overrightarrow{EX} : \overrightarrow{DX} = 2 : 1$.
$\overrightarrow{EX} = -2a + 2b$

G is the midpoint of CD.



a) Draw the vector $\overrightarrow{AO}$ and label it in terms of **a** and **b**.

$\overrightarrow{AO} = b$ → O is the midpoint between AD. So, $\overrightarrow{OD} = \overrightarrow{AO}$

b) Label all 6 sides of the hexagon in terms of **a** and **b**.

c) Find the vector $\overrightarrow{DG}$ in terms of **a** and **b**.

D is the midpoint of $\overrightarrow{DC}$.

$\overrightarrow{DG} = \frac{1}{2}(\overrightarrow{DC})$

$\overrightarrow{DG} = \frac{1}{2}(-a)$

$= -\frac{a}{2}$

d) Find the vector $\overrightarrow{DX}$ in terms of **a** and **b**

$\overrightarrow{EX} = -2a + 2b$

$\overrightarrow{DX} = \frac{1}{2}\overrightarrow{EX}$

$= \frac{1}{2}(-2a + 2b)$

$\overrightarrow{DX} = -a + b$

e) Hence, prove that O, G and X lie on a straight line.

$\overrightarrow{OG} = \overrightarrow{OD} + \overrightarrow{DG}$
$= b - \frac{a}{2}$

$\overrightarrow{GX} = \overrightarrow{GD} + \overrightarrow{DX}$
$= \frac{a}{2} + (-a + b)$
$= -\frac{a}{2} + b$

$\overrightarrow{OX} = \overrightarrow{OD} + \overrightarrow{DX}$
$= b + (-a + b)$
$= 2b - a$

$\overrightarrow{OX} = 2\overrightarrow{OG} = 2\overrightarrow{GX}$

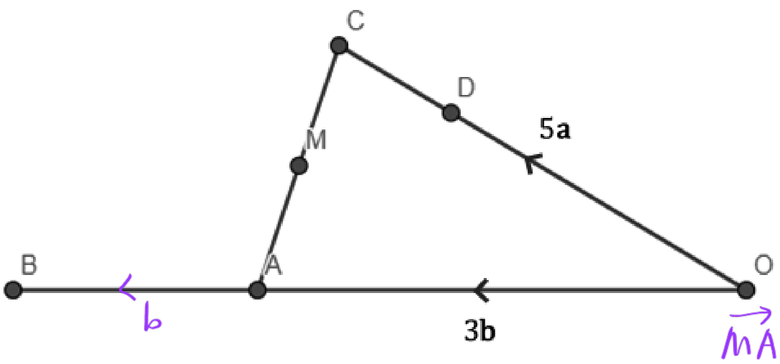
The lines $\overrightarrow{OX}$, $\overrightarrow{OG}$ and $\overrightarrow{GX}$ are parallel as they are multiple of each other. They also share a common point G, hence, O, X and G lies on a straight line.

5. The triangle AOC lies on a straight line OB.

$\overrightarrow{OC} = 5\boldsymbol{a}$ and $\overrightarrow{OA} = 3\boldsymbol{b}$

D is the point such that $\overrightarrow{OD} : \overrightarrow{DC} = 4 : 1$
M is the midpoint of $\overrightarrow{AC}$
B is the point such that $\overrightarrow{OA} : \overrightarrow{AB} = 3 : 1$

Show that D, M and B lie on the same straight line.



$\overrightarrow{MA} = \overrightarrow{CM}$

$$\overrightarrow{DM} = \overrightarrow{DC} + \overrightarrow{CM}$$
$$= \frac{1}{5} \times \overrightarrow{OC} + \frac{\overrightarrow{CA}}{2}$$
$$= \frac{1}{5} \times \overrightarrow{OC} + \frac{(\overrightarrow{CO} + \overrightarrow{OA})}{2}$$
$$= \frac{1}{5} \times 5a + \frac{(-5a + 3b)}{2}$$
$$= a - \frac{5}{2}a + \frac{3}{2}b$$
$$= -\frac{3}{2}a + \frac{3}{2}b$$

$$\overrightarrow{MB} = \overrightarrow{MA} + \overrightarrow{AB}$$
$$= -\frac{5}{2}a + \frac{3}{2}b + \frac{1}{3}(\overrightarrow{OA})$$
$$= -\frac{5}{2}a + \frac{3}{2}b + \frac{1}{3}(3b)$$
$$= -\frac{5}{2}a + \frac{3}{2}b + b$$
$$= -\frac{5}{2}a + \frac{5}{2}b$$

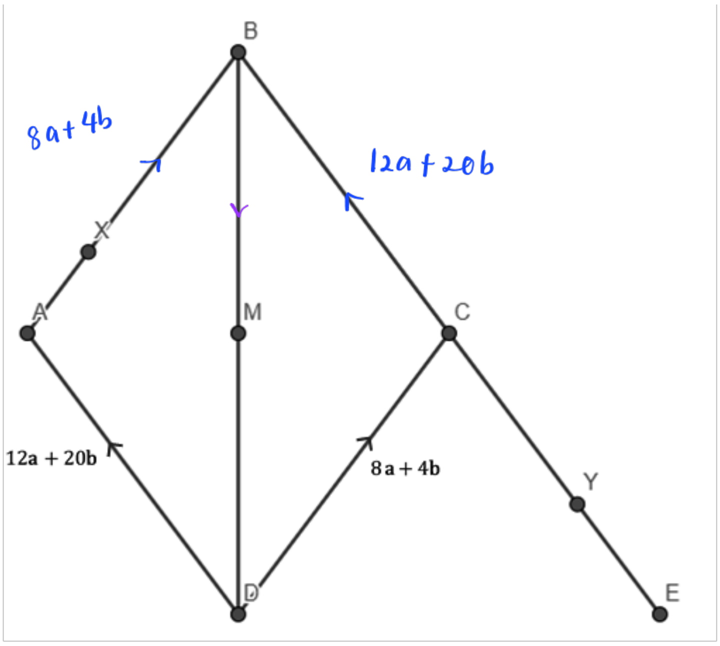
$$\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB}$$
$$= \frac{4}{5} \times (-5a) + 3b + b$$
$$= -4a + 4b$$

$\overrightarrow{DB} = \frac{8}{3}\overrightarrow{DM} = \frac{8}{5}\overrightarrow{MB}$. As $\overrightarrow{DB}$, $\overrightarrow{DM}$ and $\overrightarrow{MB}$ are multiple of each other, this proves that they are parallel. They also share a common point, M which means point D, M and B lie on the same straight line.

6. ABCD is a parallelogram.

$\overrightarrow{DA} = 12\mathbf{a} + 20\mathbf{b}$ and $\overrightarrow{DC} = 8\mathbf{a} + 4\mathbf{b}$.
X lies on the line $\overrightarrow{AB}$ such that $\overrightarrow{AX} : \overrightarrow{XB} = 1 : 3$
M is the midpoint of $\overrightarrow{DB}$
$\overrightarrow{CE}$ is an extension of $\overrightarrow{BC}$
Y lies on the line $\overrightarrow{CE}$ such that $\overrightarrow{CY} = -\frac{1}{2}\overrightarrow{DA}$.

Prove that X, M and Y are collinear.



$\overrightarrow{XM} = \overrightarrow{XB} + \overrightarrow{BM}$

$= \frac{3}{4}(\overrightarrow{AB}) + \frac{\overrightarrow{BD}}{2}$

$= \frac{3}{4}(8a + 4b) + \frac{(\overrightarrow{BC} + \overrightarrow{CD})}{2}$

$= \frac{3}{4}(8a + 4b) + \frac{(-12a - 20b - (8a + 4b))}{2}$

$= \frac{3}{4}(8a + 4b) + \frac{(-20a - 24b)}{2}$

$= 6a + 3b - 10a - 12b$

$= -4a - 9b$

$\overrightarrow{MY} = \overrightarrow{MB} + \overrightarrow{BY}$

$= \overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CY}$

$= (10a + 12b) + (-12a - 20b) - \frac{1}{2}(12a + 20b)$

$= 10a + 12b - 12a - 20b - 6a - 10b$

$= -8a - 18b$

$\overrightarrow{XY} = \overrightarrow{XB} + \overrightarrow{BY}$

$= 6a + 3b + (-12a - 20b) - 6a - 10b$

$= 6a + 3b - 12a - 20b - 6a - 10b$

$= -12a - 27b$

$\overrightarrow{XY} = 3\overrightarrow{XM} = \frac{3}{2}\overrightarrow{MY}$. As $\overrightarrow{XY}$, $\overrightarrow{XM}$ and $\overrightarrow{MY}$ are multiple of each other, this proves that they are parallel. They also share a common point M. Hence, X, M and Y are collinear.