Higher Check In - 9.03 Plane vector geometry

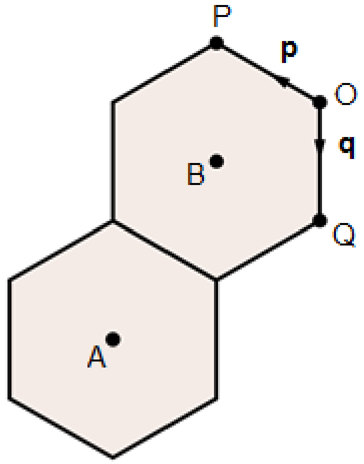
In questions 1 and 2, $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$.

1. Work out $2\mathbf{a} + \mathbf{b} - \mathbf{c}$.

2. Work out $\mathbf{b} + \frac{1}{2}(\mathbf{a} + \mathbf{c})$.

3. Point A has coordinates (7, 4). Point B has coordinates (11, –4). Work out $\overrightarrow{AB}$.

4. A and B are the centres of the two regular congruent hexagons shown below. Express $\overrightarrow{AB}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.



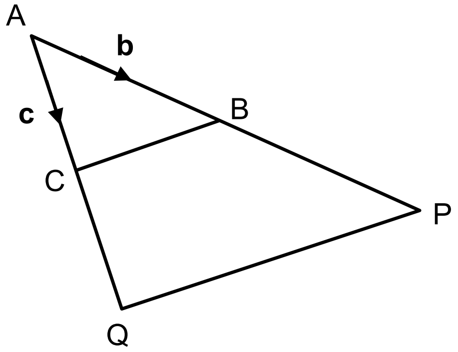
Not to scale

5. A and B are the points such that $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$. M is the midpoint of line AB. Find the vector $\overrightarrow{OM}$.

6. Explain how you can determine that the vectors $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3.5 \end{pmatrix}$ are parallel without needing to draw them.

7. $\overrightarrow{OA} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Show that A, B and C are points on a single straight line.

8. ABC and APQ are triangles. B is the midpoint of AP and C is the midpoint of AQ. $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$. Prove that $\overrightarrow{BC}$ is parallel to $\overrightarrow{PQ}$.



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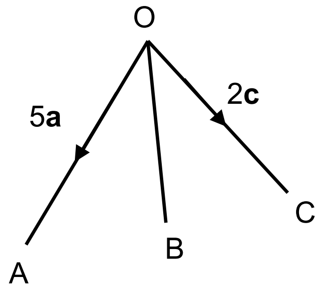
9. $\mathbf{a} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$. Work out $\mathbf{c}$ if $3(\mathbf{a} + \mathbf{c}) = \mathbf{b}$.

10. On the diagram below, $\overrightarrow{OA} = 5\mathbf{a}$ and $\overrightarrow{OC} = 2\mathbf{c}$. B is the point on line AC such that AB : BC = 3 : 2. Express $\overrightarrow{OB}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.



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Extension

The diagonals of a parallelogram bisect each other (i.e. the diagonals cross so that they meet at their midpoints: this cuts each diagonal into two parts of equal length). Use vector methods to prove that this is always true for a parallelogram.

Answers

1. $2\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}-\begin{pmatrix}3\\-3\end{pmatrix}=\begin{pmatrix}6\\-2\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}-\begin{pmatrix}3\\-3\end{pmatrix}=\begin{pmatrix}6+2-3\\-2+5+3\end{pmatrix}=\begin{pmatrix}5\\6\end{pmatrix}$

2. $\begin{pmatrix}2\\5\end{pmatrix}+\frac{1}{2}\left(\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}3\\-3\end{pmatrix}\right)=\begin{pmatrix}2\\5\end{pmatrix}+\frac{1}{2}\begin{pmatrix}6\\-4\end{pmatrix}=\begin{pmatrix}2\\5\end{pmatrix}+\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$

3. $\overrightarrow{AB}=\begin{pmatrix}11\\-4\end{pmatrix}-\begin{pmatrix}7\\4\end{pmatrix}=\begin{pmatrix}4\\-8\end{pmatrix}$

4. $-\mathbf{p}-2\mathbf{q}$

5. $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-7\\-5\end{pmatrix}-\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}-10\\-3\end{pmatrix}$.

   $\overrightarrow{OM}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AB}=\begin{pmatrix}3\\-2\end{pmatrix}+\begin{pmatrix}-5\\-1.5\end{pmatrix}=\begin{pmatrix}-2\\-3.5\end{pmatrix}$.

   Alternatively, the coordinates of A and B could be used to find the midpoint (–2, –3.5) which could then be converted to a position vector.

6. Since $-2\begin{pmatrix}-2\\3.5\end{pmatrix}=\begin{pmatrix}4\\-7\end{pmatrix}$ one of the vectors is a scalar multiple of the other and therefore they are parallel.

7. $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}1\\4\end{pmatrix}-\begin{pmatrix}-3\\2\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$. $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}3\\5\end{pmatrix}-\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$. Since $\overrightarrow{AB}=2\overrightarrow{BC}$ then $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$. As they share a common point (B), the three points ABC must be on a single straight line.

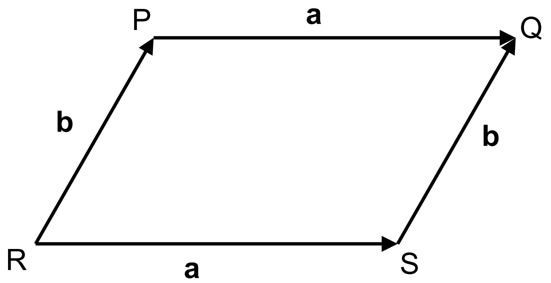
8. $\overrightarrow{BC}=\overrightarrow{BA}+\overrightarrow{AC}=-\mathbf{b}+\mathbf{c}=\mathbf{c}-\mathbf{b}$. $\overrightarrow{AP}=2\mathbf{b}$ and $\overrightarrow{AQ}=2\mathbf{c}$, so $\overrightarrow{PQ}=\overrightarrow{PA}+\overrightarrow{AQ}=2\mathbf{c}-2\mathbf{b}$. Since $\overrightarrow{PQ}=2\overrightarrow{BC}$ then $\overrightarrow{BC}$ is parallel to $\overrightarrow{PQ}$.

9. Dividing $3\left(\begin{pmatrix}-4\\3\end{pmatrix}+\mathbf{c}\right)=\begin{pmatrix}-3\\-9\end{pmatrix}$ by 3 gives $\begin{pmatrix}-4\\3\end{pmatrix}+\mathbf{c}=\begin{pmatrix}-1\\-3\end{pmatrix}$ so $\mathbf{c}=\begin{pmatrix}-1\\-3\end{pmatrix}-\begin{pmatrix}-4\\3\end{pmatrix}=\begin{pmatrix}3\\-6\end{pmatrix}$.

10. $\overrightarrow{OA}=5\mathbf{a}$. $\overrightarrow{AC}=2\mathbf{c}-5\mathbf{a}$. $\overrightarrow{AB}=\frac{3}{5}\overrightarrow{AC}$. $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=5\mathbf{a}+\frac{3}{5}(2\mathbf{c}-5\mathbf{a})=2\mathbf{a}+\frac{6}{5}\mathbf{c}$.

Extension

The opposite sides of a parallelogram are equal in length and parallel, therefore they can be represented by the same vector.



Not to scale

Diagonal $\overrightarrow{PS} = \mathbf{a} - \mathbf{b}$ and diagonal $\overrightarrow{RQ} = \mathbf{a} + \mathbf{b}$. Since these diagonals are not parallel, they will cross at a point. If X is the point where they cross, we can describe $\overrightarrow{RX}$ by $\overrightarrow{RX} = r(\overrightarrow{RQ}) = r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$ and also by $\overrightarrow{RX} = \mathbf{b} + s(\overrightarrow{PS}) = \mathbf{b} + s(\mathbf{a} - \mathbf{b}) = \mathbf{b} + s\mathbf{a} - s\mathbf{b}$ where r and s are fractions. Since these describe the same journey, they are equal so $\mathbf{b} + s\mathbf{a} - s\mathbf{b} \equiv r\mathbf{a} + r\mathbf{b}$.

Comparing the coefficients of **a** on each side we get $s = r$ (this tells us they cross the same fraction along each diagonal, although we do not yet know what fraction.)

Comparing the coefficients of **b** on each side we get $1 - s = r$. Since $s = r$ we have $1 - r = r$ so $1 = 2r$ and $r = \frac{1}{2}$.

Therefore the point where the two diagonals meet is halfway along each diagonal i.e. the diagonals bisect each other.

GCSE (9–1)

MATHEMATICS

Assessment Objective	Qu.	Topic	R	A	G
AO1	1	Carry out arithmetic with vectors			
AO1	2	Carry out arithmetic with vectors			
AO1	3	Find a vector			
AO1	4	Use vectors in geometric arguments			
AO1	5	Use vectors to find a midpoint			
AO2	6	Use vector methods to show two vectors are parallel			
AO2	7	Use vectors to prove three points are on a single straight line			
AO2	8	Use vectors in a geometric proof			
AO3	9	Solve a problem involving vectors			
AO3	10	Use vectors in geometric arguments			

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