GCSE Maths – Number

Exact Values and Surds

Notes

WORKSHEET

Exact Values

Some values cannot be written exactly. An example of this is the numerical value of π, $3.14159265...$ However many digits we write, it will not be exact because π has infinite digits after the decimal point. Therefore, we write it as the symbol π to denote the **exact value**. When we use an exact value like π in our calculations, we leave it as the symbol.

Similarly, we may use **fractions** in our calculations that cannot be expressed as a **terminating decimal**. An example of this is $\frac{1}{7}$, which is a **recurring decimal**. If we were to use the number in its decimal form, we would at some point round a digit, which loses **accuracy**. Therefore, we keep the fractions as they are, so that we keep the exact value throughout the calculation, and then round at the end if appropriate.

Example: Calculate the value of $7\pi + 4\pi$
We do not need to convert π to its decimal form - instead, we leave it as its ***exact value****.* *π can be treated almost the same as x in algebra - it just represents a number.* *In this calculation, we just collect the multiples of π and add their total. This gives us $7\pi + 4\pi = 11\pi$.* *For the final answer, we could leave it in exact value as 11π, or we could use a calculator to find the value of 11π, which is 34.558 $(3\ d.p.)$ Note, this value of 34.558 is not the exact answer since we have rounded.*

Calculating with Exact Fractions

When we calculate with exact fractions, we have to keep the fractions in their **original form**, rather than converting them to a decimal or percentage. This makes our calculations **more accurate**, particularly when dealing with fractions that do not convert to a **terminating decimal**, because we do not need to **round** them.

We need to know how to add, subtract, multiply and divide fractions.

To add and subtract fractions, we have to ensure that all fractions have the **same denominator** (number of the bottom). This might require **multiplying** the **numerator** and **denominator** of one particular fraction by the same number.

Example: Calculate $1\frac{2}{7} + \frac{1}{5}$. Leave the answer as an exact fraction.

1. To work this out, we need to keep the fractions in their exact forms, rather than converting to a rounded decimal form.

   The mixed fraction, $1\frac{2}{7}$, can be converted to $\frac{9}{7}$, since it is essentially the same as $\frac{7}{7} + \frac{2}{7}$.

2. To add two fractions with different denominators, we need to find the **lowest common multiple** of both denominators.

   The lowest common multiple of 7 and 5 is 35. We need to multiply each fraction by another fraction, which is equal to 1, to ensure that the denominators are the same.

$$\frac{9}{7} \times \frac{5}{5} = \frac{45}{35}$$

   To keep the fraction the same, whatever we multiply the denominator by, we must also multiply the numerator by.

$$\frac{1}{5} \times \frac{7}{7} = \frac{7}{35}$$

3. We can now add these fractions together because they have the same denominator. We leave the answer in its exact fraction form.

$$\frac{45}{35} + \frac{7}{35} = \frac{52}{35}$$

Example: Calculate $\frac{8}{9} - \frac{2}{3}$

1. We follow a similar approach as before, finding the lowest common multiple of both denominators.

   In this case, it is 9. This means that the first fraction does not need to be changed.

   The second fraction needs to be multiplied to give a denominator of 9. We must multiply the numerator and denominator by the same number.

$$\frac{2}{3} \times \frac{3}{3} = \frac{6}{9}$$

2. The fractions can now be subtracted.

$$\frac{8}{9} - \frac{6}{9} = \frac{2}{9}$$

   Our final answer is left in exact fraction form.

To multiply exact fractions, we simply multiply the numerators of each fraction, and the denominators of each fraction.

Example: Calculate $\frac{5}{6} \times \frac{4}{7}$

We keep both fractions in their exact form and multiply the numerators and denominators:

$$5 \times 4 = 20$$
$$6 \times 7 = 42$$

$$\frac{5}{6} \times \frac{4}{7} = \frac{20}{42}$$

Check if the final fraction can be simplified:

$$\frac{20}{42} = \frac{10}{21}$$

Dividing fractions is slightly different. When one fraction is divided by another, we flip the numerator and denominator of the second fraction, then perform a multiplication operation.

Example: Calculate $\frac{8}{11} \div \frac{2}{5}$

1. Flip the numerator and denominator of the second fraction $\frac{2}{5}$.

   This gives us:

   $$\frac{5}{2}$$

2. Carry out the multiplication operation.

   $$\frac{8}{11} \times \frac{5}{2} = \frac{40}{22}$$

3. Simplify the final fraction if possible, leaving it in its exact form.

   $$\frac{8}{11} \times \frac{5}{2} = \frac{40}{22} = \frac{20}{11}$$

Surds (Higher Only)

Just like certain fractions and π, the square roots of many numbers cannot be written exactly. An example of this is $\sqrt{3}$, which is $1.732050808\ldots$. Instead of writing this decimal, we use $\sqrt{3}$ in our calculations to ensure we are using the **exact value**. When the square root of a number is left in this form because it **cannot be written exactly**, it is called a **surd**.

When performing surd calculations, we can almost treat them like algebra terms such as x.

Example: What is $3\sqrt{3} + 5\sqrt{3}$?
Don't get put off by the surds! *Imagine that we substitute $\sqrt{3}$ for x. We can then write $3x + 5x$, which is $8x$.* *Now we change x back to $\sqrt{3}$.* *Final answer is $8\sqrt{3}$.*

Example: What is $3\sqrt{5} + 5\sqrt{3} + 7\sqrt{5}$?
*This question requires adding together surds that are the **same**. We can only add surds that have the **same number under the square root**.* *$3\sqrt{5}$ and $7\sqrt{5}$ can be added to get $10\sqrt{5}$, but we cannot add $5\sqrt{3}$ because it is a **different surd**.* *Alternatively, this can be thought of as $3x + 5y + 7x$, where $x = \sqrt{5}$ and $y = \sqrt{3}$. The x terms can be summed.* *Final answer:* $3\sqrt{5} + 5\sqrt{3} + 7\sqrt{5} = 10\sqrt{5} + 5\sqrt{3}$ *(left in exact value form rather than rounded decimal form)*

Simplifying Surds (Higher Only)

We can sometimes make surds simpler (smaller) by looking at the **number under the square root**. This requires finding a **factor** that is a **square number**.

For example, take the surd $\sqrt{24}$. We know that 4 and 6 are **factors** of 24. This means that this surd can also be written as $\sqrt{4 \times 6}$, which can be split up into $\sqrt{4} \times \sqrt{6}$. Now, 4 is a **square number**, and its **square root can be written exactly** as 2. Therefore, we can simplify this in $2 \times \sqrt{6}$, or just $2\sqrt{6}$.

Example: Simplify the surd $\sqrt{48}$

First, think of factors of 48 *that are* ***square numbers****.*

16 *and* 3 *are factors, and* 16 *is a square number.*

We can write $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3}$.

The square root of 16 *is exactly* 4, *so this can be simplified.*

$\sqrt{16} \times \sqrt{3} = 4 \times \sqrt{3} = 4\sqrt{3}$.

Example: Simplify the surd $5\sqrt{72}$

First, think of factors of 72.

36 *and* 2 *are factors, and* 36 *is a square number.*

Therefore, this can be written as $5 \times \sqrt{36 \times 2} = 5 \times \sqrt{36} \times \sqrt{2}$

This can be simplified to $5 \times 6 \times \sqrt{2} = 30\sqrt{2}$

Final answer: $30\sqrt{2}$

Simplifying the surds may be required before we can add or subtract.

Example: What is $5\sqrt{12} + 6\sqrt{48}$?

$5 \times \sqrt{12}$ *is the same as* $5 \times \sqrt{4 \times 3}$, *or* $5 \times \sqrt{4} \times \sqrt{3}$.
This can be simplified to $5 \times 2 \times \sqrt{3}$, *which is* $10\sqrt{3}$.

$6 \times \sqrt{48} = 6 \times \sqrt{16 \times 3} = 6 \times \sqrt{16} \times \sqrt{3}$.
This can then be simplified to $6 \times 4 \times \sqrt{3} = 24\sqrt{3}$.

We can now add these terms together.

$10\sqrt{3} + 24\sqrt{3} = 34\sqrt{3}$

Multiplication and Division of Surds (Higher Only)

Multiplying and dividing surds is slightly different. When we **multiply surds**, we simply multiply the **numbers** under the **square root** sign.

If the number under the square root is the **same** in each surd, then multiplication will give an **integer** answer. For example, $\sqrt{6} \times \sqrt{6}$ is the same as $(\sqrt{6})^2$, so the square root is cancelled out, and the answer is 6.

Alternatively, if the numbers under the square root are **different**, then we **multiply** the numbers and **keep the square root**. To work out $\sqrt{7} \times \sqrt{5}$, we can write it as $\sqrt{7 \times 5}$, which is $\sqrt{35}$. Be sure to check if the surds can be **simplified** before multiplying them. For example, $\sqrt{7} \times \sqrt{8}$ can be written as $\sqrt{7} \times 2\sqrt{2}$. The numbers under the square root are multiplied, giving $2\sqrt{14}$.

Example: Calculate $\sqrt{11} \times \sqrt{12}$

Before multiplying the surds, check to see if they can be ***simplified****.*

$$\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}.$$

Now the calculation looks like $\sqrt{11} \times 2\sqrt{3}$*:*

$$\sqrt{11} \times 2\sqrt{3} = 2\sqrt{11 \times 3} = 2\sqrt{33}$$

Example: Calculate $5\sqrt{6} \times 2\sqrt{5}$

The numbers under the square root and the numbers outside the square root are ***treated separately*** *here.*

We separately calculate 5×2 *(numbers outside the square root), and* $\sqrt{6} \times \sqrt{5}$ *(numbers inside the square root).*

This gives us $10\sqrt{30}$*.*

When **dividing surds**, we follow a similar approach. The numbers under the square root are treated separately from the numbers outside the square root. For example, to calculate $\sqrt{10} \div \sqrt{5}$, we can keep the square root and work out the division. This gives us $\sqrt{2}$.

Example: Calculate $27\sqrt{15} \div 9\sqrt{5}$

The numbers under the square root and the numbers outside the square root are ***treated separately*** *here.*

We separately calculate $27 \div 9 = 3$ *(numbers outside the square root), and* $\sqrt{15} \div \sqrt{5} = \sqrt{3}$ *(numbers inside the square root).*

This gives us $3\sqrt{3}$*.*

Remember that we might be able to simplify the surd before performing the division, which makes it easier!

Example: Calculate $4\sqrt{54} \div \sqrt{3}$

*First, let's **simplify** the first surd.*

$$4\sqrt{54} = 4\sqrt{9 \times 6} = 4\sqrt{9} \times \sqrt{6} = 4 \times 3 \times \sqrt{6} = 12\sqrt{6}.$$

*Then continue with the rest of the calculation. Don't get put off if there's no number outside the square root, it can be treated as **1**!*
This gives $12 \div 1 = 12$.

$$12\sqrt{6} \div \sqrt{3} = 12\sqrt{2}$$

Rationalising the Denominator (Higher Only)

Rationalising the denominator means manipulating a fraction so that we **remove a surd** from the **denominator** (bottom of the fraction), making it an **integer**.

Take the fraction $\frac{6}{\sqrt{5}}$. We want to remove the surd from the bottom to make it a **rational denominator**. To do this, we need to multiply it by another fraction. To make the denominator $\sqrt{5}$ an **integer**, it needs to be multiplied by $\sqrt{5}$, because $(\sqrt{5})^2 = 5$. Therefore, the denominator of the multiplying fraction is $\sqrt{5}$.

What is the numerator of the second fraction? We can't **alter the original fraction** in any way or change its value, so all we want to do is **multiply it by 1**. To ensure that the second fraction is equal to 1, the numerator and denominator need to be the same. In this case, the numerator is also $\sqrt{5}$.

$$\frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{6 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{6\sqrt{5}}{5}$$

We can leave the surd in the numerator because we just need to ensure that the **denominator is an integer**.

Example: Rationalise the denominator of the fraction $\frac{7}{\sqrt{11}}$

*To make $\sqrt{11}$ an integer, it needs to be **squared** (multiplied by itself). Therefore, the **denominator** of the fraction that we need to multiply the original fraction by is $\sqrt{11}$.*

*As the numerator and denominator need to be the **same** (to make sure we are **multiplying the original fraction by 1**), the numerator of the new fraction is also $\sqrt{11}$.*

$$\frac{7}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}} = \frac{7\sqrt{11}}{\sqrt{11} \times \sqrt{11}} = \frac{7\sqrt{11}}{11}$$

Be sure to check whether the surd or fraction can be simplified too!

Example: Rationalise the denominator of the fraction $\frac{8\sqrt{12}}{2\sqrt{10}}$

1. First, we should simplify the fractions.

   The numbers outside the square root can be written as $\frac{4\sqrt{12}}{\sqrt{10}}$.

2. Then we can simplify the surds.

$$\frac{4\sqrt{12}}{\sqrt{10}} = \frac{4\sqrt{4 \times 3}}{\sqrt{10}} = \frac{4\sqrt{4} \times \sqrt{3}}{\sqrt{10}} = \frac{8\sqrt{3}}{\sqrt{10}}$$

3. Now we can **rationalise the denominator**.

   We can multiply

$$\frac{8\sqrt{3}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{8\sqrt{3} \times \sqrt{10}}{\sqrt{10} \times \sqrt{10}} = \frac{8\sqrt{30}}{10}$$

The fraction can then be ***simplified*** *again, giving us* $\frac{4\sqrt{30}}{5}$.

If we have the denominator in the form for example: $2\sqrt{3} + 2$, what do we multiply it with to make it an integer?

We use difference between two squares, multiply the expression with itself but reverse the sign. It becomes:

$$(2\sqrt{3} + 2)(2\sqrt{3} - 2) = (2\sqrt{3})^2 - 2^2$$
$$= 4(3) - 4 = 12 - 4 = 8$$

Example: Rationalise the denominator of the fraction $\frac{2\sqrt{5}}{\sqrt{7}+3}$

Rationalise the denominator using the difference of two squares:

We can multiply $\frac{2\sqrt{5}}{\sqrt{7}+3} \times \frac{\sqrt{7}-3}{\sqrt{7}-3}$.

$$\frac{2\sqrt{5}(\sqrt{7} - 3)}{(\sqrt{7})^2 - 3^2} = \frac{2\sqrt{35} - 6\sqrt{5}}{7 - 9} = \frac{6\sqrt{5} - 2\sqrt{35}}{2}$$

Exact Values - Practice Questions

For all the questions, write your answer in the exact from.

1. Calculate $1\frac{1}{2} - \frac{4}{7}$

2. Calculate $\frac{9}{11} \div \frac{5}{3}$

3. Calculate $4\frac{2}{3} \times \frac{2}{3}$

Surds (Higher Only) - Practice Questions

4. Calculate $2\sqrt{3} + \sqrt{5} + 3\sqrt{3} + 4\sqrt{5}$

5. Simplify $6\sqrt{12}$.

6. Calculate $10\sqrt{3} \times 2\sqrt{27}$

7. Simplify $\frac{15\sqrt{10}}{3\sqrt{2}}$.

8. Rationalise the denominator for the fraction $\frac{9\sqrt{2}}{18\sqrt{3}}$.

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.