GCSE Maths – Geometry and Measures

Rotation, Reflection, Translation and Enlargement

Notes

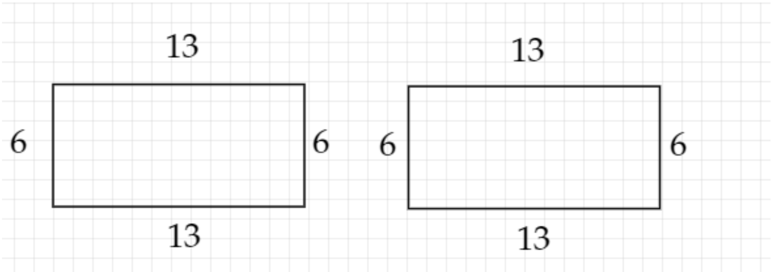
WORKSHEET

Congruent and similar shapes

Congruent shapes

Two shapes are **congruent** if they are exactly the same shape and have the same size.

Congruent shapes have identical properties –their **side lengths** and **angles** are the same.
The following two shapes are congruent:



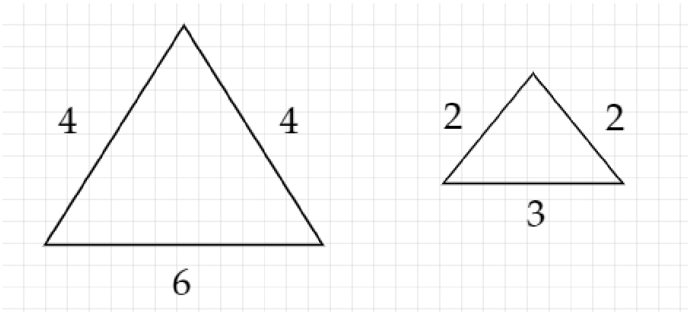
These are both rectangles and have the same length and width measurements.

Congruent shapes may be **rotations** of the same shape.

Similar shapes

Two shapes are similar if they are the same shape but one may be an **enlargement** of the other.

Although lengths may differ for similar shapes, their **angles** are the same.
The following two shapes are similar:



They are both triangles and the shape on the left is an enlargement of the shape on the right. The scale factor is 2 because all side lengths are multiplied by 2.

Triangles are only similar if all their **corresponding angles are equal**. This is a fact which can be used to answer questions about **similar triangles**.

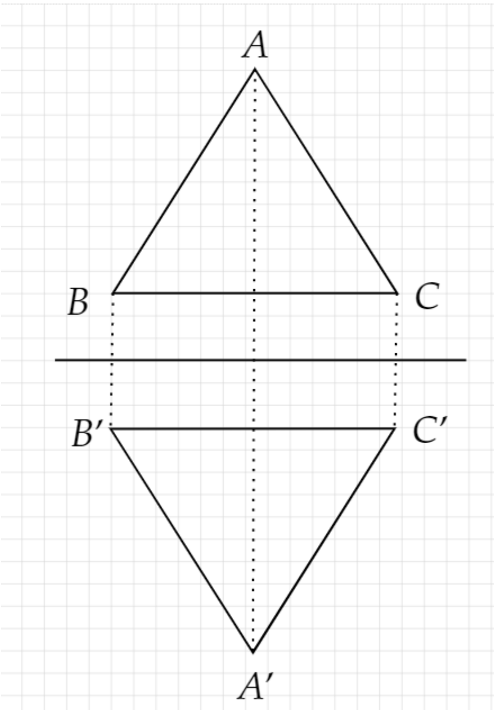
Transformation

A transformation involves **changing the size and/or position** of a shape.
The following are the transformations we consider:

- Reflection
- Translation
- Enlargement
- Rotation

Reflection

Reflection is creating a **mirror image** of a shape across a **line**. Every point of the shape is drawn on the **opposite side** of the line, exactly the **same distance** from the line as the original point.



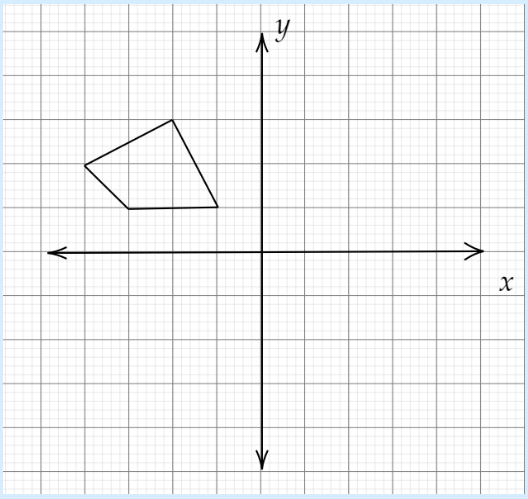
Original shape: ABC
Reflected image: $A'B'C'$

The line in which the shape has been reflected is called the **mirror line**.

- The distance from A to the mirror line is equal to the distance from A' to the mirror line.

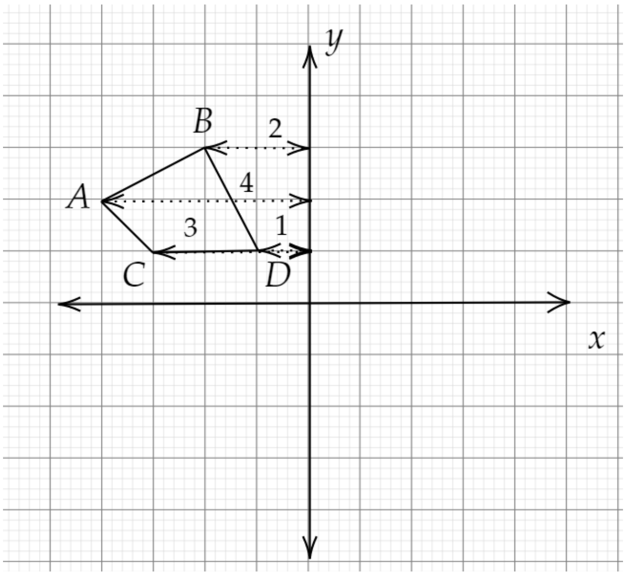
  This is the same with all the points on the triangle.

Example: Reflect the given shape in the line $x = 0$.

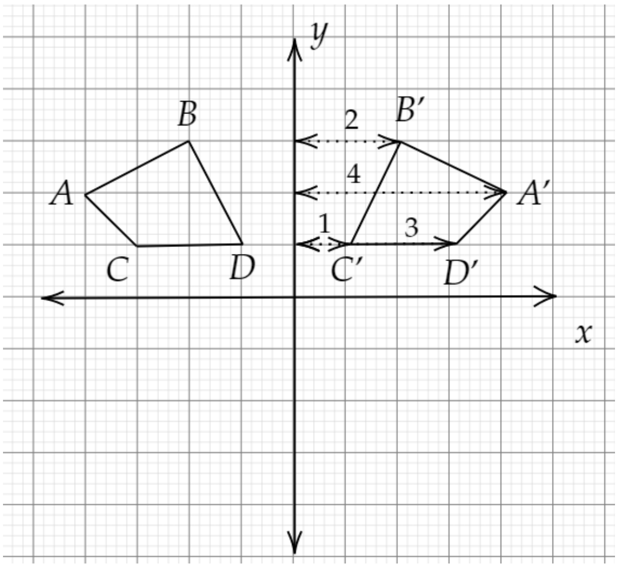


The line $x = 0$ is the line which corresponds to the y $axis$.

Firstly, label all the points and find the distances from the points to the mirror line.

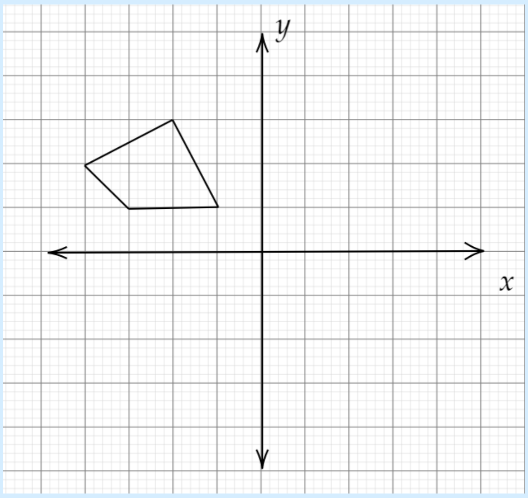


Now draw the new image, marking the points the same distance away from the mirror line.

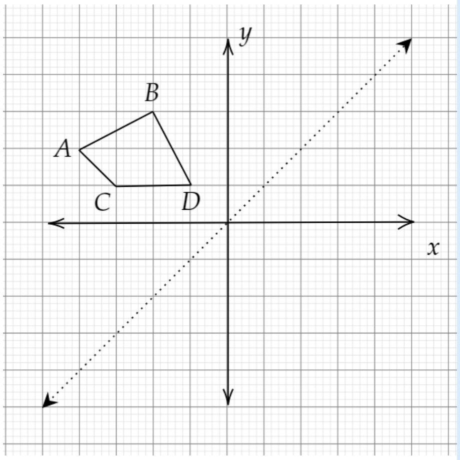


The mirror line can be diagonal as well, but the approach stays the same.

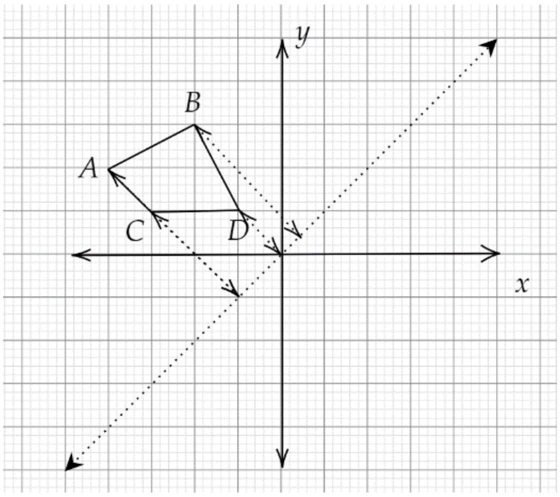
Example: Reflect the following shape in the line $y = x$



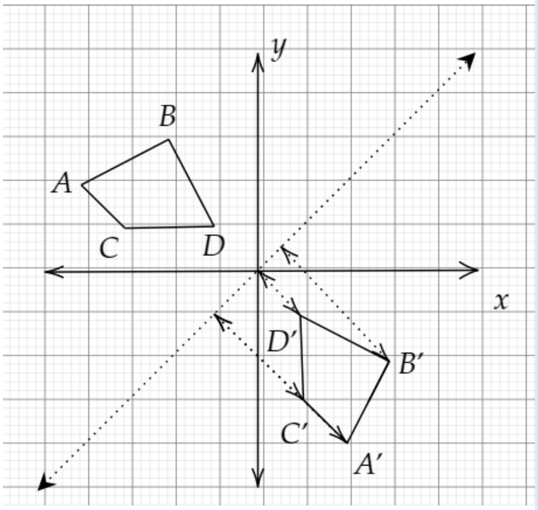
The line $y = x$ is the mirror line.



Firstly, label all the points and find the distances from the points to the mirror line.



Now draw the new image, marking the new points the same distance away from the mirror line.



Translation

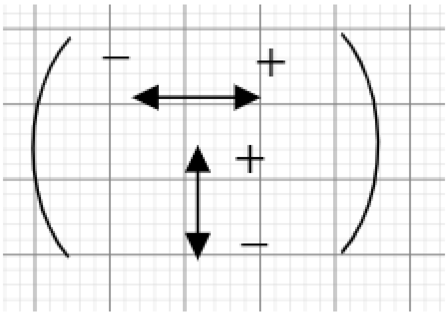
Translations involves moving the shape in **different directions** without making any changes to the size of the original shape.

Column vectors can be used to describe the movement of a shape. A column vector is given in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

where x describes the **horizontal** movement and y describes the **vertical** movement.

- When x is positive, the shape moves x units in the positive x direction (to the right).
- When x is negative, the point moves x units in the negative x direction (to the left).
- When y is positive, the point moves y units in the positive y direction (upwards).
- When y is negative, the point moves y units in the negative y direction (downwards).
- If 0 is in place of x or y, the shape does not move parallel to the horizontal or vertical axis, respectively.



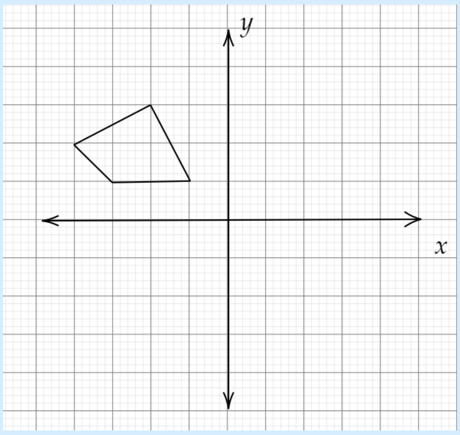
Examples

$\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ → Move the shape 5 squares right and 3 squares down.

$\begin{pmatrix} -13 \\ 6 \end{pmatrix}$ → Move the shape 13 squares left and 6 squares up.

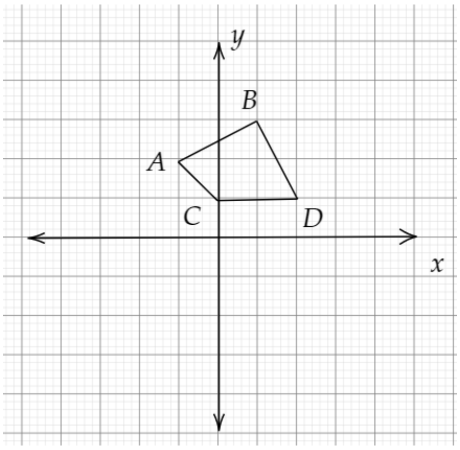
$\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ → Move the shape 2 squares left and 4 squares down.

Example: Translate the following shape by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

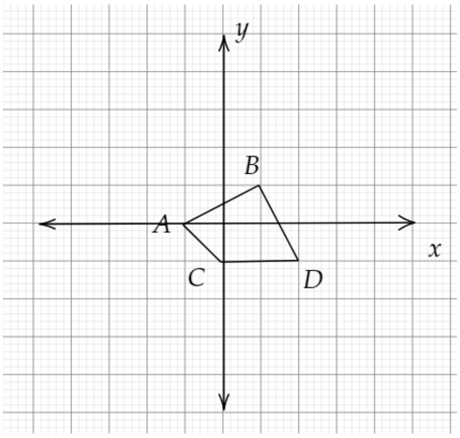


The vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ means:

"Move 3 squares to the right and move 2 squares down."



1. *So, first each point of the shape moves 3 squares to the right:*



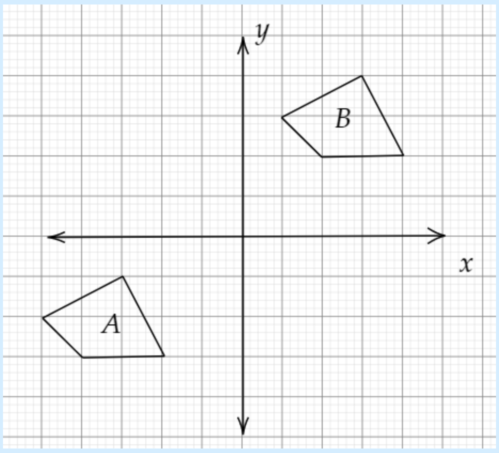
2. *Now each point moves 2 squares down:*

That is the translation complete.
The order in which you complete the translation does not matter.

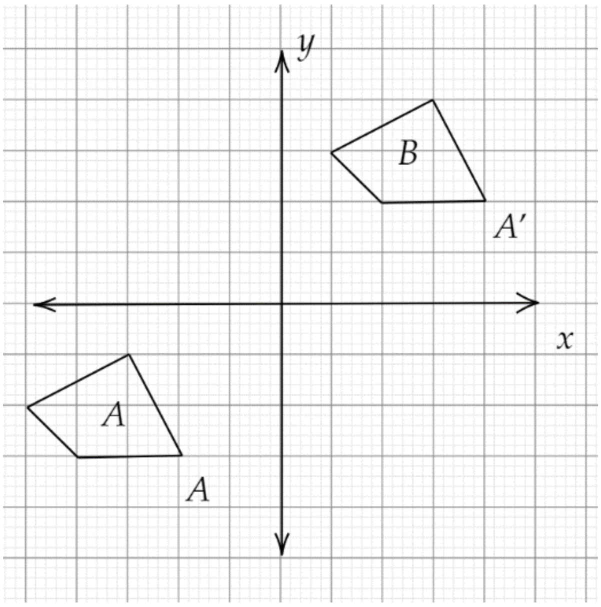
If we would have done shifting it down 2 first and then shifting it right 3, we would have ended in the same place. This is a property of vectors.

As an extra activity you can try different vectors on the same shape, and then do it all again with the same vectors but in different order. You will find that the shape ends up in the same place, regardless of whether the shape is first moved horizontally or vertically.

Example: Shape A has been translated to the position of shape B. Find the translation vector.



1. *Take one point on shape A and mark it. Then mark its corresponding point on shape B.*

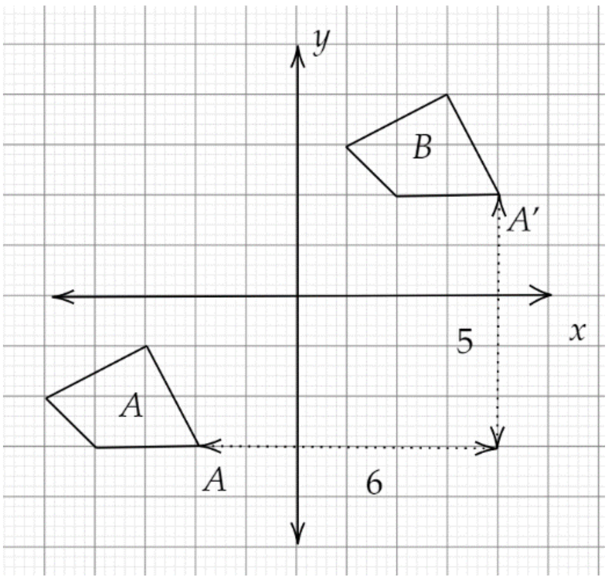


2. *Count how many units horizontally and vertically point A has moved.*

   Write the horizontal distance as the top number of the vector and the vertical distance as the bottom number of the vector.

$$\begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

   As the shape is moving right and up, both the numbers are positive.

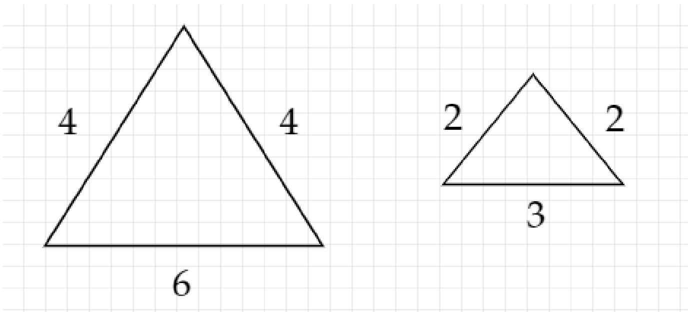


Enlargement

Enlargement is **changing the size** of a shape by a given amount. This is not always increasing the size; depending on the given amount, it could involve a reduction in size.

The amount by which we enlarge the shape is called the **scale factor**.

The following is a reference to a previously seen example:



The above example can show two enlargements:

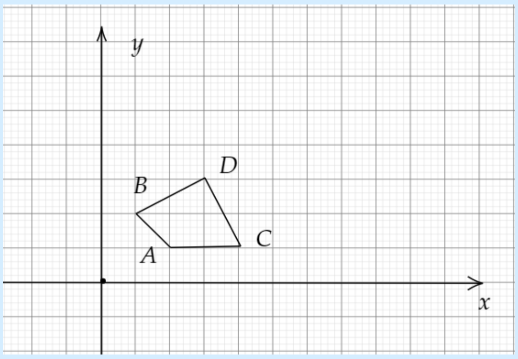
- The shape on the left is an enlargement of the shape on the right by a scale factor of 2.
- The shape on the right is an enlargement of the shape on the left by a scale factor of $\frac{1}{2}$.

The size of the scale factor incdicates whether the enlargement will show an increase in size or a reduction in size:

- If the **scale factor is > 1** then we are making the shape **bigger**.
- If the **scale factor is < 1** then we are making the shape **smaller**.

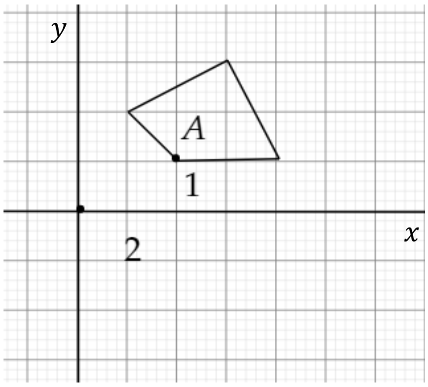
The scale factor tells us the value by which we are enlarging the shape, however it does not tell us the position of the transformed shape. Therefore, if we want to see where the enlarged shape is positioned, we need a **centre of enlargement**, which determines the position of the transformed shape.

Example: Enlarge the shape by a scale factor of 3, with the centre of enlargement as the origin.



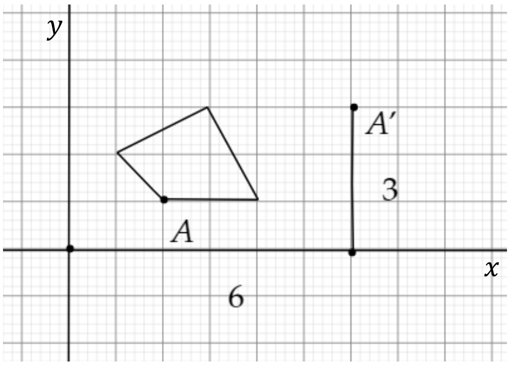
1. Firstly, take one point on the shape.

   Here, I have chosen point A.



2. Find the horizontal and vertical distance from the centre of enlargement to the point.

   From the origin, point A is 2 units across and 1 unit up.



3. Multiply both these values by the scale factor.

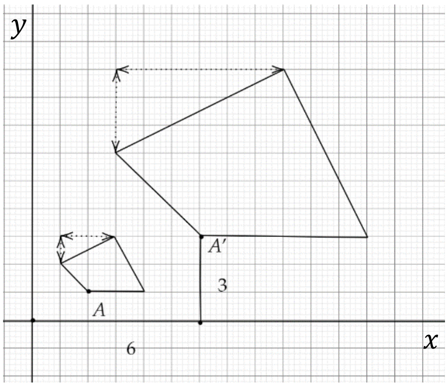
   $Scale\ factor = 3$

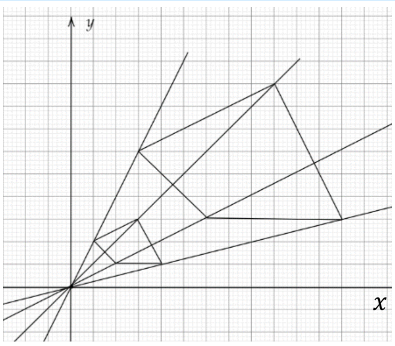
   $2 \times 3 = 6$

   $1 \times 3 = 3$ → *A' has vector* $\binom{6}{3}$

Now the position given by the vector is the new point A'. This means A' is 6 units across and 3 units up.

From here we can work out what the rest of the shape will look like. We multiply the length of each side by the scale factor 3. For the diagonal sides, we can find the horizontal and vertical distance and then multiply them by 3. Join the remaining points together.



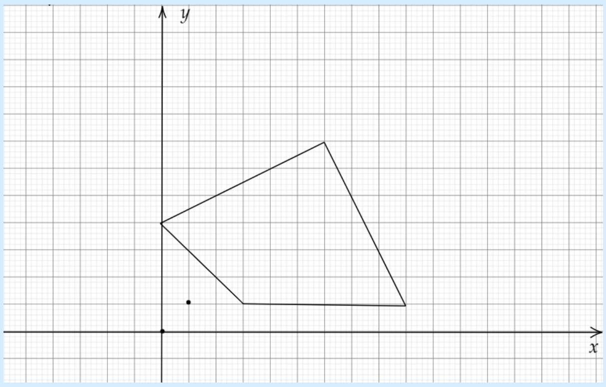


*If you don't know the centre of enlargement, draw lines joining corresponding points in the shape and its image. The point where all these lines meet is the **centre of enlargement**.*

Fractional scale factors

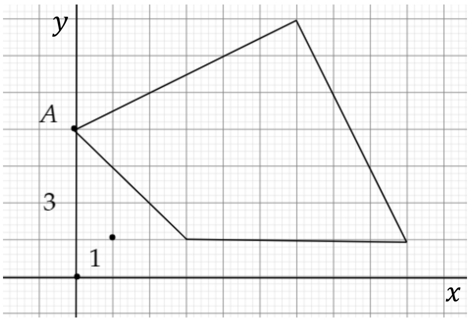
For fractional scale factors, we follow the **same steps** as in the above example. This time there is a possibility of the shape getting **smaller** if the scale factor is less than 1.

Example: Enlarge the following shape by a scale factor of $\frac{1}{2}$ with the centre of enlargement $(1,1)$, as marked on the graph.



1. Firstly, take one point on the shape.

   Here, I have chosen point A as shown.



2. Find the horizontal and vertical distance from the centre of enlargement to the point.

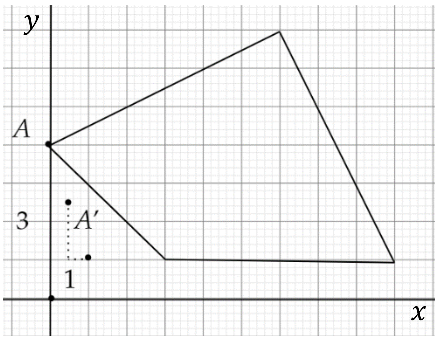
   From the centre of enlargement $(1,1)$, point A is 1 unit left (in the negative x direction) and 3 units up.

3. Multiply both these values by the scale factor.

   $Scale\ factor = 0.5$

   $3 \times 0.5 = 1.5$
   $-1 \times 0.5 = -0.5$

   → *A' has vector $\binom{1.5}{-0.5}$ from $(1,1)$*

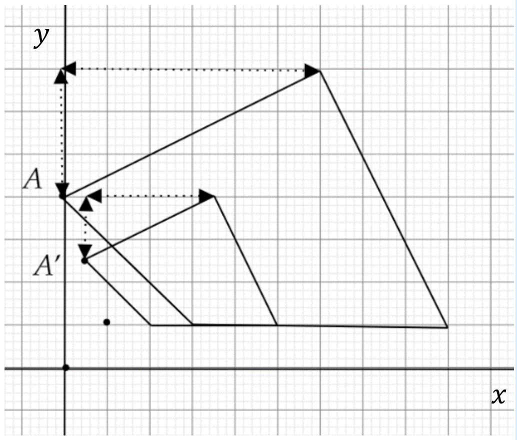


4. Starting from the centre of enlargement, use the vector $\binom{1.5}{-0.5}$ to find where point A' will be.

   From here we can work out the rest of the transformed shape:

   - *Multiply the length of each side by scale factor 0.5.*
   - *For the diagonal sides, find the horizontal and vertical distance to the centre of enlargement and multiply them by scale factor 0.5.*

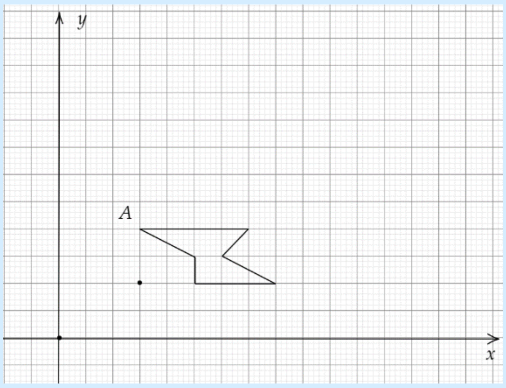
   Join the two remaining points together.



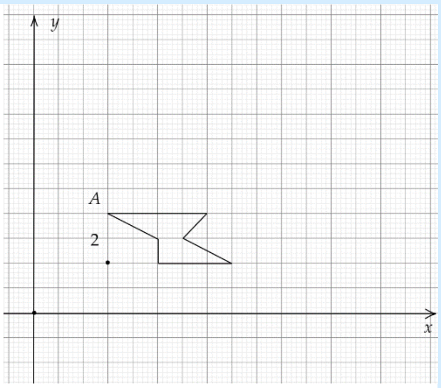
Negative scale factors of enlargement (Higher)

The scale factor of enlargement can be **negative**. However, the method for drawing the enlargement does not change.

Example: Enlarge the following shape by a scale factor of -2 with the centre of enlargement $(3,2)$, as shown on the graph.



1. Firstly, take one point on the shape.
   Here, I have chosen point A as shown.



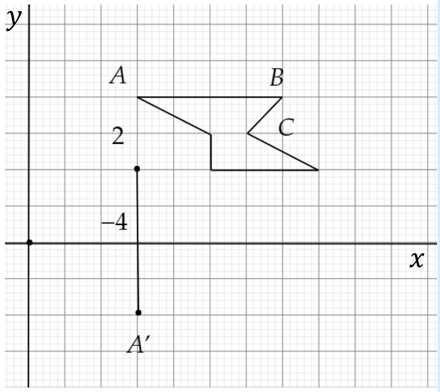
2. Find the horizontal and vertical distance from the centre of enlargement to the point.

   From the centre of enlargement $(3,2)$, point A has moved 0 units in the x direction and 2 units up in the positive y direction.

3. Multiply both these values by the scale factor.

$$Scale\ factor = -2$$
$$2 \times (-2) = -4$$

→ *A' has vector $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ from $(3,2)$.*



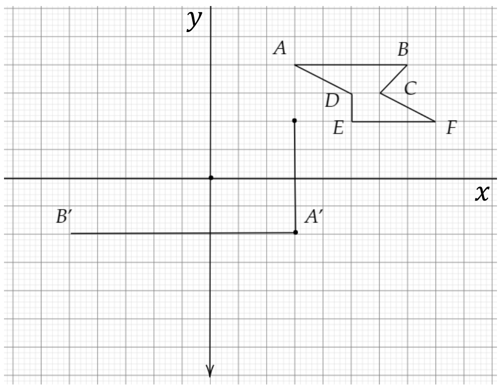
4. Starting from the centre of enlargement, use the above vector to find where point A' will be.

   -4 represents moving down 4 squares.

   From here we can work out what the rest of the shape will look like:

   - *We multiply the length of each side by scale factor -2.*

$$AB = 4$$
$$A'B' = 4 \times (-2) = -8$$



We can continue this for the other sides:

- *EF is 2 units right of the centre $(3,2)$ and has a length of 3.*

$$2 \times (-2) = -4$$
$$3 \times (-2) = -6$$

  So, $E'F'$ has length of 6 and is 4 units from the centre $(3,2)$ in the negative x direction.

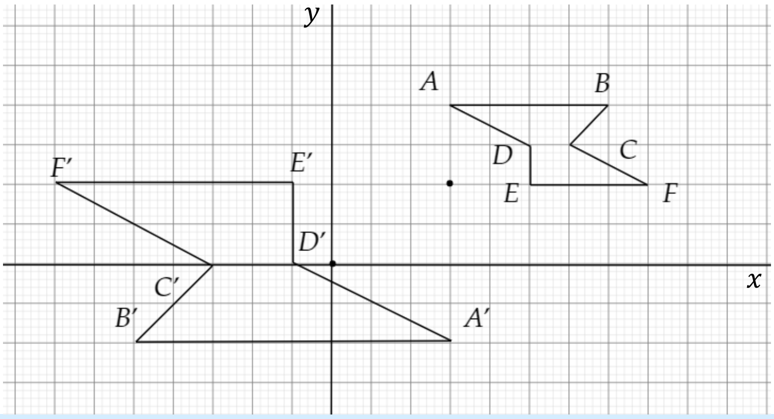
- *ED has length of 1.*

$$E'D' = 1 \times (-2) = -2$$

  So $E'D'$ has length -2 and is positioned below the centre of enlargement (due to the negative sign). This means we extend E' 2 units down to find position D'.

- *D' and A' can be joined together on the transformed image.*

- *C and D are one unit apart in the x direction. Therefore, $C'D'$ will be -2 units apart. The negative sign here can be ignored, what's important is that it tells us $C'D'$ are 2 units apart in the transformed shape.*



When finding the vector between the centre of enlargement and the point on the shape, always **start from the centre** going towards the point. Do not find the vector from the point to the centre as that would give an incorrect vector.

Rotation

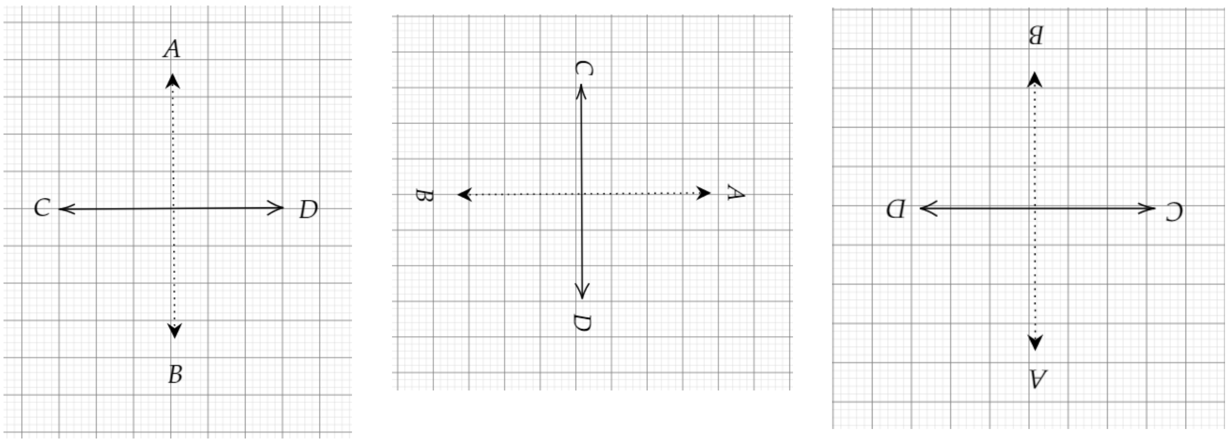
In a rotation transformation, the shape is turned around a fixed point.

Centre of rotation: The fixed point around which the shape is turned.

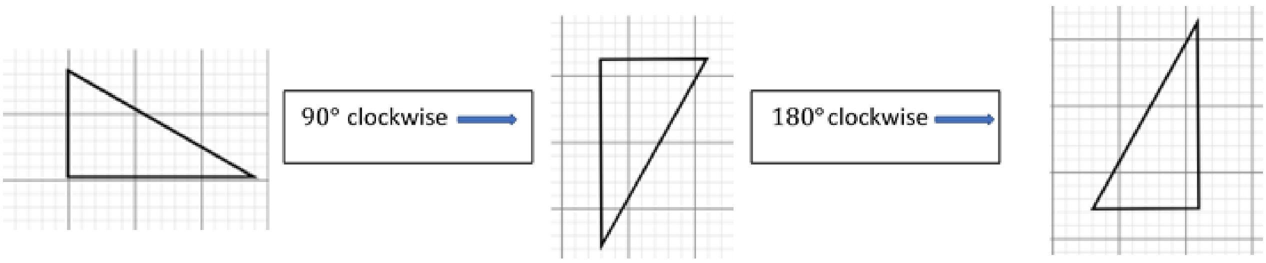
In reflection, the only piece of information to perform the transformation was the mirror line. However, if performing rotation, we need more information. We need to know:

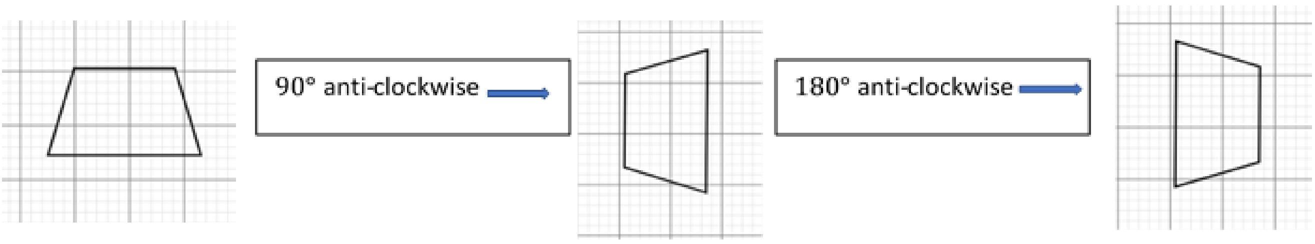
- The **co-ordinates** of the centre of rotation.
- The **angle** by which the object is rotated.
- The **direction** of rotation (clockwise or anti-clockwise).

The diagram below is rotated $90°$ clockwise every time. In the 3rd diagram it has been rotated $180°$ clockwise.



Similarly, with shapes:

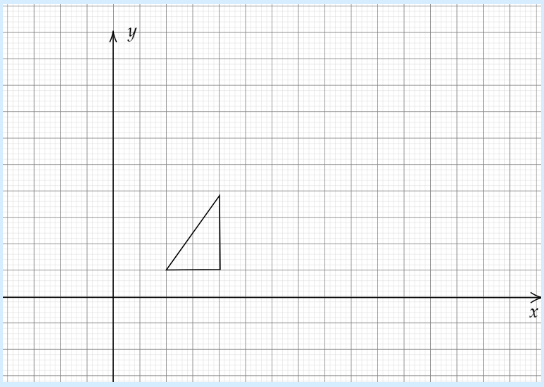




Rotation with centre of rotation is tricky. We have to consider the position of the shape on the graph and accordingly rotate it.

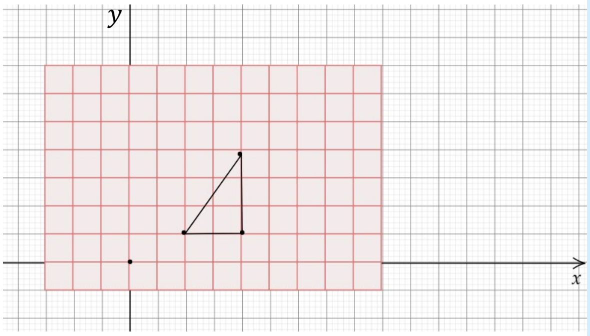
The use of **tracing paper** can be used to help with rotations. The example below shows how to complete a rotation using tracing paper.

Example: Rotate the following shape $90°$ anti-clockwise around the origin.

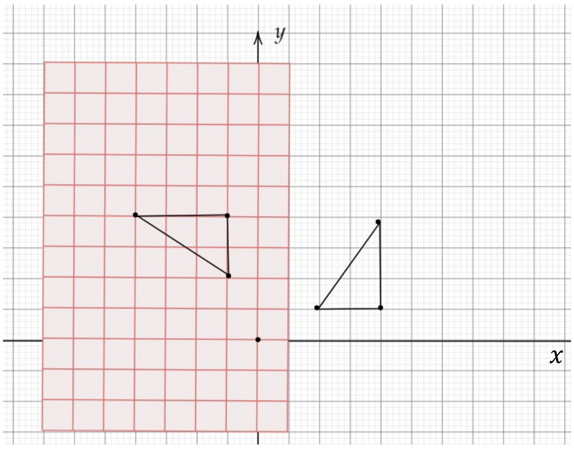


1. Place the tracing paper onto the graph.

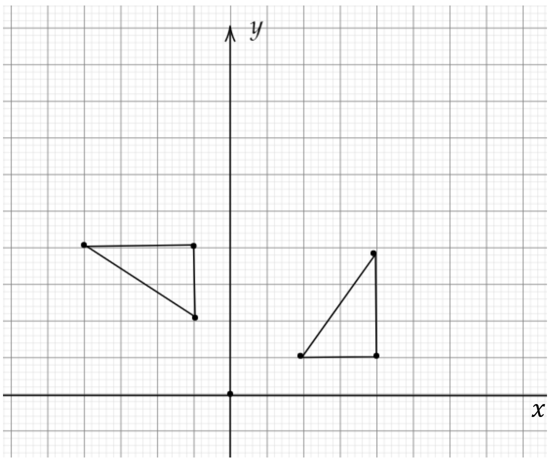
2. Trace the original shape on the tracing paper and mark the centre of rotation.



3. Put your writing pen down on the centre of rotation and hold the paper in that position. Keeping your writing pen on the centre, spin the tracing paper $90°$ anti-clockwise.

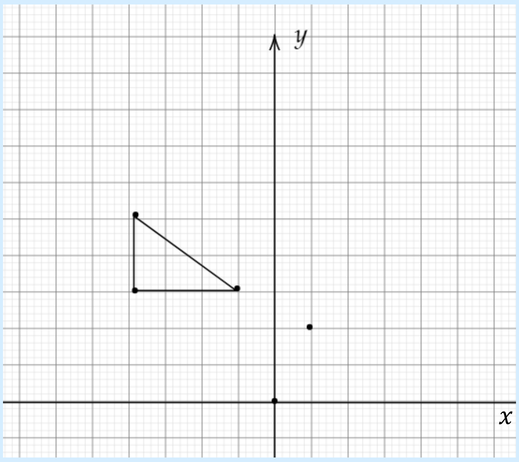


4. This is the final image. Copy it onto the graph paper.

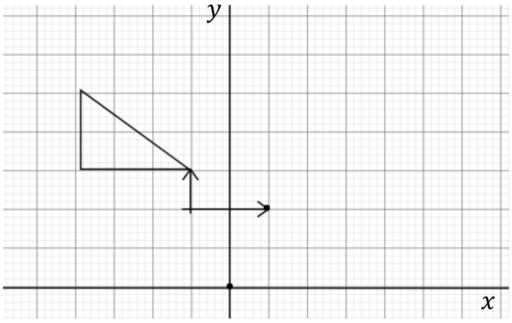


You can choose which way you want to tackle the question.
The following example is how to perform rotation without tracing paper.

Example: Rotate the following shape $90°$ clockwise with centre of rotation $(1,2)$.



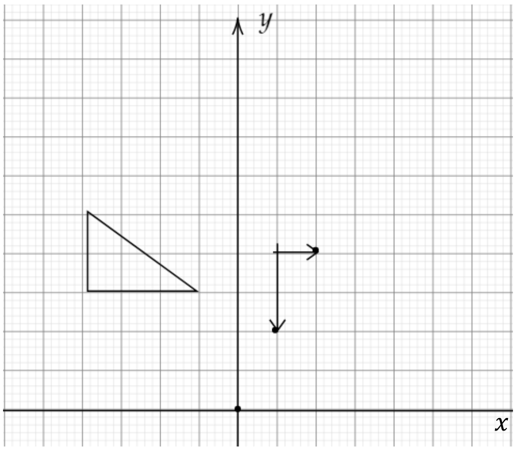
1. Choose the point on the shape that is closest to the centre.



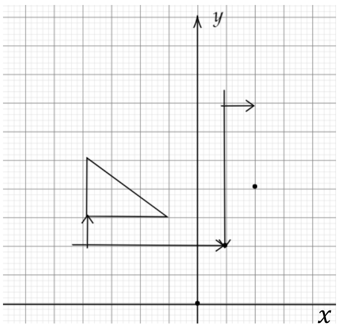
2. Mark the vector that takes this point to the centre like the following.

3. Rotate this vector around the point.

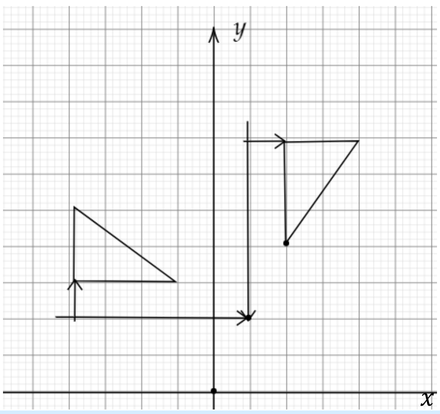
   The end of the vector on the centre stays still and it is the other point that moves.



   Do the same step with another point:



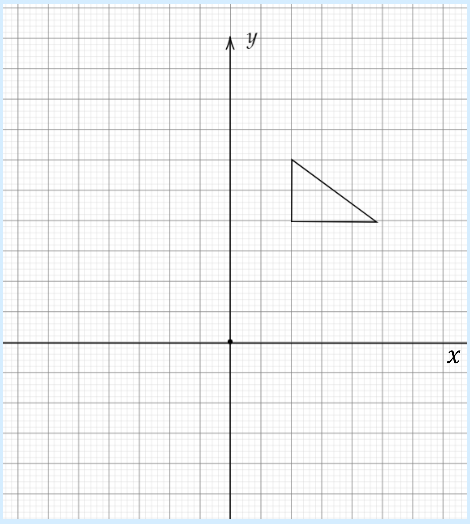
4. Now, the rotated image can be formed by connecting the points.



Combined Transformations (Higher)

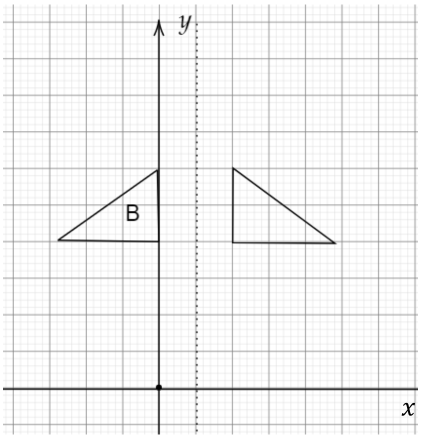
One transformation can be followed by another, for example: a reflection followed by translation. Questions with multiple transformations are tackled in the same way as we tackled the previous examples.

Example: Reflect the following shape in the line $x = 1$. Label the new shape B. Reflect shape B in the line $y = 3$. Label this shape C. Rotate shape C about the origin clockwise $180°$.

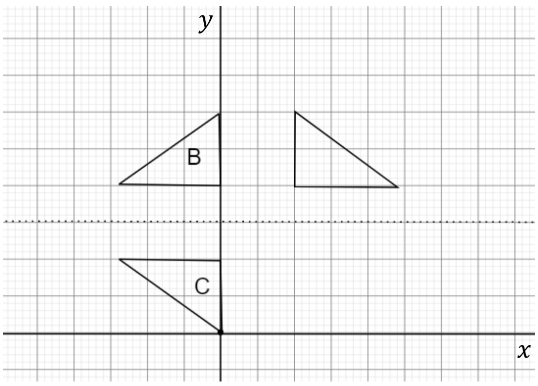


It is important to do all the transformations in the order given.

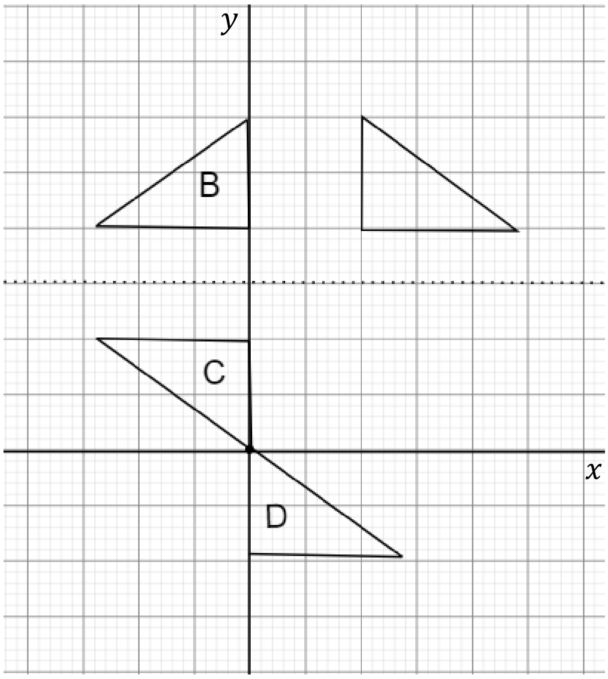
1. Reflection in the line $x = 1$



2. Reflect shape B in line $y = 3$.



3. Rotate shape C around the origin $180°$ clockwise.



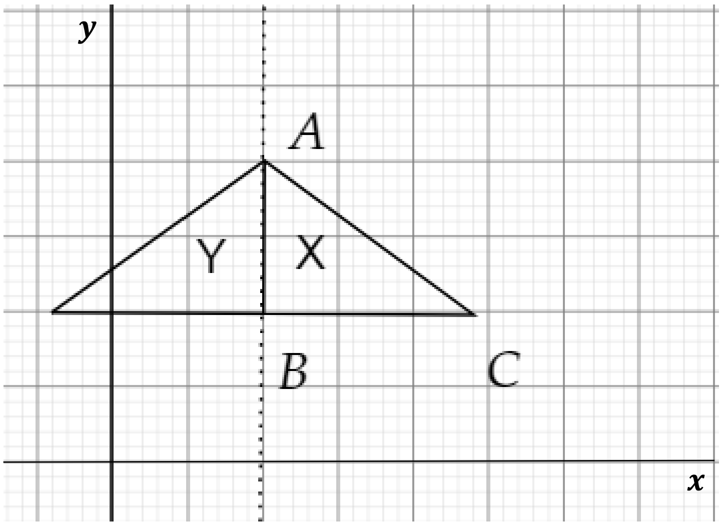
If we look at the final image shape D, it is a translation of the original shape by the vector $\binom{-2}{-6}$.

Invariance

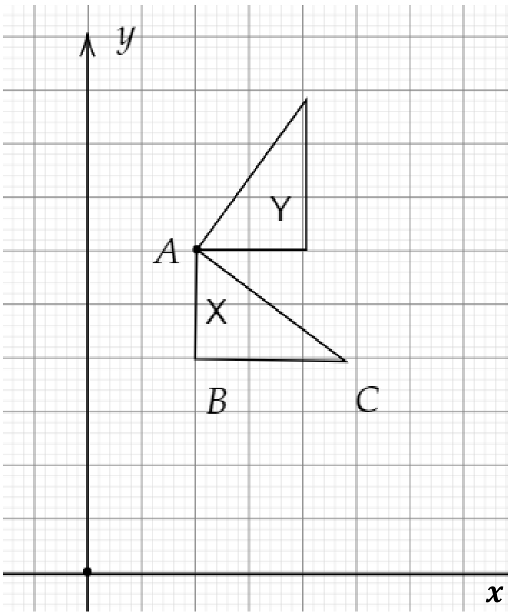
Invariant points are those which are **unaffected by the transformation**. They stay fixed after the transformation has taken place. There can be both **invariant points** and **invariant lines**.

The following are examples of some invariant points and invariant lines.

- Shape X is reflected in the line $x = 2$, to make shape Y. Line AB stays fixed which means it is invariant.



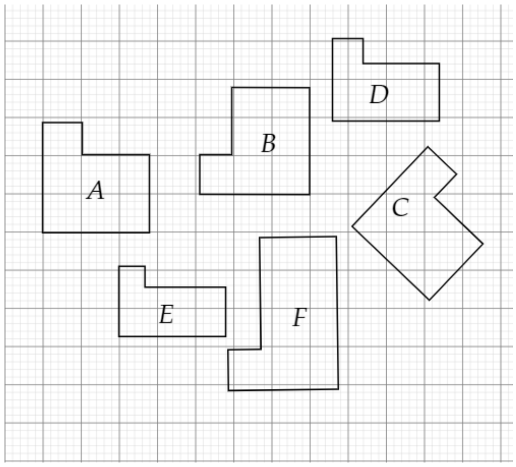
- Shape X is rotated $90°$ anti-clockwise about the point $(2,6)$, Point A is invariant. Notice, point A is also the centre of rotation.



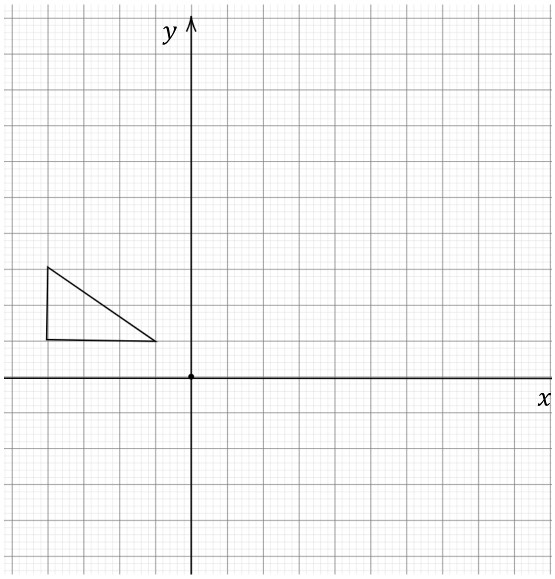
Invariance can be spotted by just looking at the points that remain **fixed** throughout the transformations.

Rotation, Reflection, Translation and Enlargement – Practice Questions

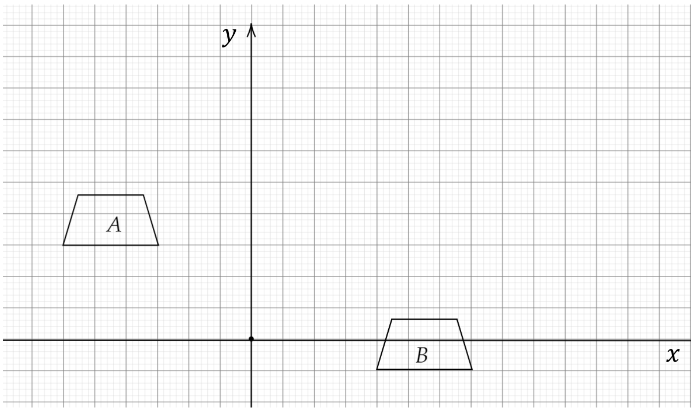
1. From the following shapes, identify all the shapes that are congruent.



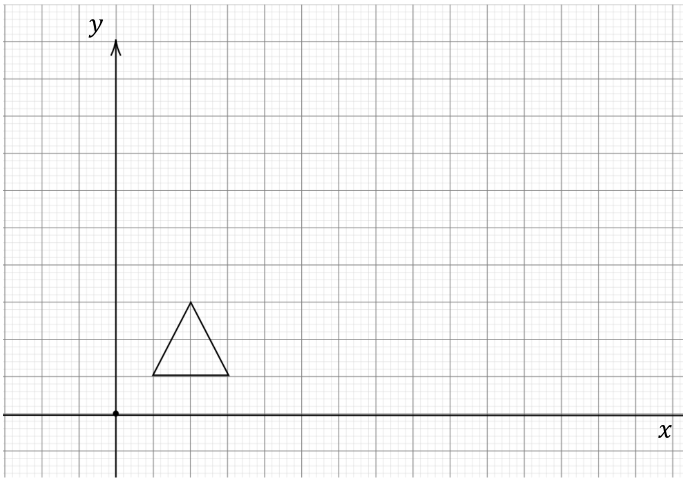
2. Reflect the following shape in the line $y = x + 1$.



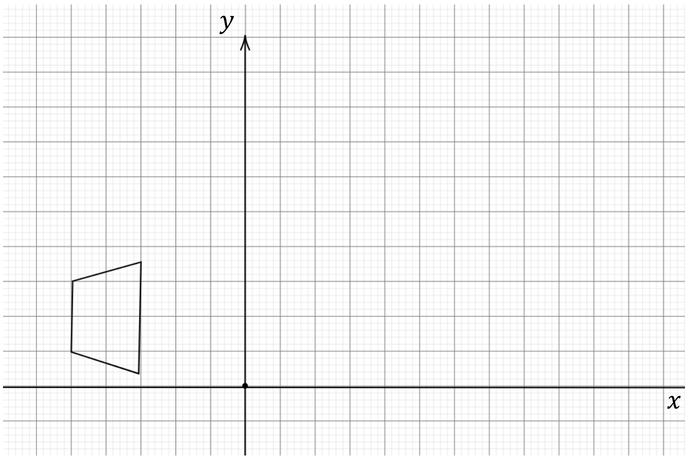
3. Describe the transformation that maps shape A to shape B. Then, translate shape B by the vector $\begin{pmatrix} -5 \\ 7 \end{pmatrix}$.



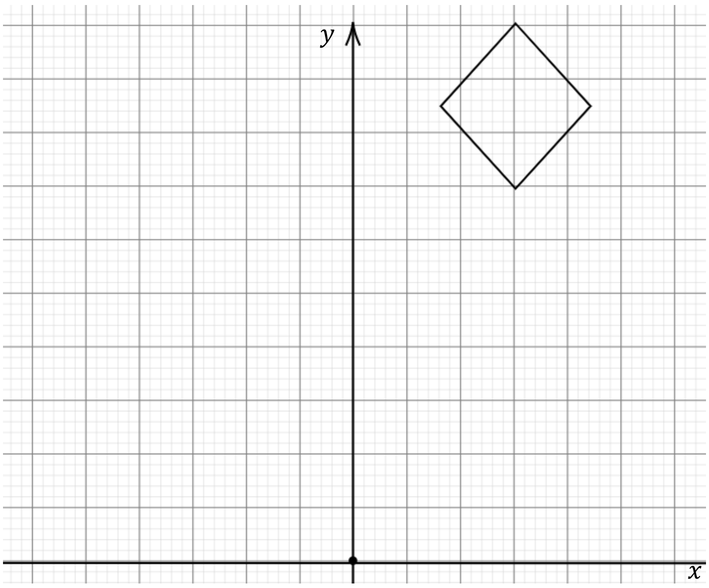
4. Enlarge the following shape by a scale factor of 4, centre of enlargement is $(2,3)$.



5. Transform the shape by the scale factor $\frac{3}{2}$. Centre of enlargement $(0,1)$. After drawing the final image, check your answer by mapping the centre of enlargement from the image you have drawn.

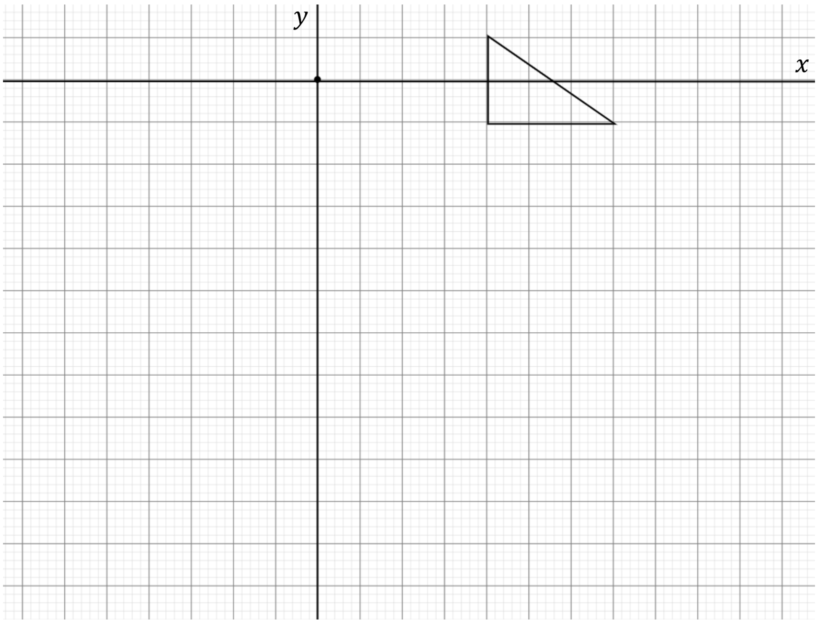


6. Rotate the following shape $90°$ anti-clockwise. Centre of rotation $= (0,4)$.

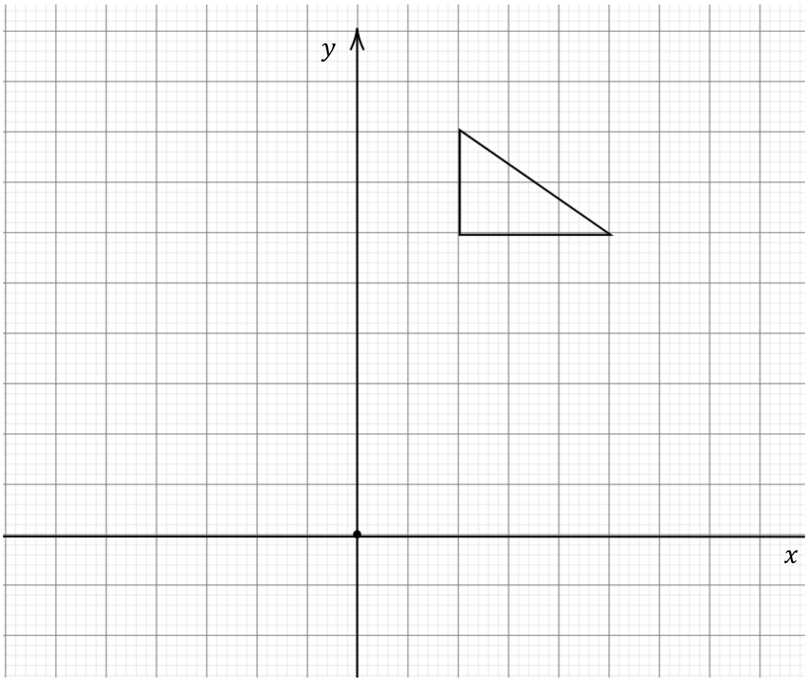


Practice Questions (Higher)

7. Enlarge the following shape by the scale factor $-\frac{3}{2}$ with centre $(-3,-2)$.



8. Rotate the following shape $270°$, clockwise, labelling it B, centre of rotation being $(0,5)$. Reflect B in the line $y = x$. Find the transformation that would map the initial shape to the final image. Are there any invariant points? If no, describe a transformation for the final image that would create 2 invariant points.



Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.