GCSE Maths – Geometry and Measures

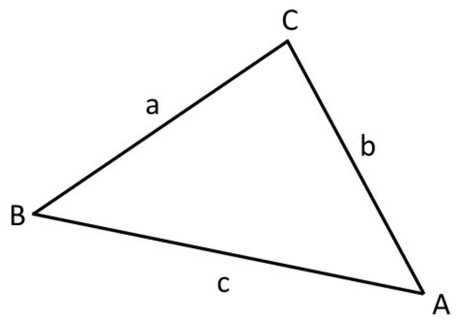
Sine and Cosine Rules and Area of a Triangle (Higher Only)

Notes

WORKSHEET

Sine Rule

The sine rule describes the **relationship** between the **lengths of sides** of a triangle and their **opposite angles**. To understand the sine rule, we first need to know how to **label** the sides and angles of any triangle:



Each **side** is labelled with a **lower-case letter**, whilst the **opposite angle** to each side is labelled with the same letter, but as a **capital**.

The sine rule is:

$$\frac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \frac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \frac{\boldsymbol{c}}{\sin \boldsymbol{C}}$$

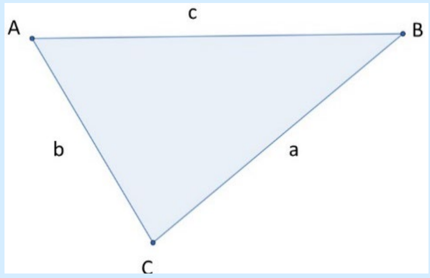
Alternatively, it can also be written as:

$$\frac{\sin \boldsymbol{A}}{\boldsymbol{a}} = \frac{\sin \boldsymbol{B}}{\boldsymbol{b}} = \frac{\sin \boldsymbol{C}}{\boldsymbol{c}}$$

The form of the sine rule that we use depends on if we are trying to find lengths or angles. Using the correct one will make rearranging the equation much easier:

- If we want to calculate angles, we need the $\sin A$, $\sin B$ or $\sin C$ to be the numerators.
- If we are trying to find the sides, we need a, b or c to be the numerators.

Example: In the following triangle angle A is $60.1°$, length a is 5.6 cm and length b is 4 cm. Use the sine rule to calculate angle B.



1. Choose the correct form of the sine rule.

   We are trying to find angle B and we are given the values of angle A and length a so we use the sine rule of the form $\frac{\sin A}{a} = \frac{\sin B}{b}$.

2. Substitute the values you know into the formula.

$$\frac{\sin 60.1}{5.6} = \frac{\sin B}{4}$$

3. Solve the equation to find angle B.

   *Multiply both sides by 4 to **remove the denominator** on the right:*

$$\sin B = 4 \times \frac{\sin 60.1}{5.6}$$

   *Use the **inverse sine button** $(sin^{-1}\ )$ to work out the angle.*

$$B = \sin^{-1}\left(4 \times \frac{\sin 60.1}{5.6}\right) = \mathbf{38.3°}$$

Example: In a triangle, angle B is $43.6°$, angle C is $67.5°$ and length c is 2.4 cm. Calculate length b.

1. Choose the correct form of the sine rule.

   We are given values of b/B and c/C so we need a sine rule involving these measurements. To calculate length, we use the sine rule in this form:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

2. Substitute the values you know into the formula.

$$\frac{b}{\sin 43.6} = \frac{2.4}{\sin 67.5}$$

3. Solve the equation to find length b.

   Rearrange the equation to get b on its own.
   To do this, multiply both sides by $\sin 43.6$:

$$b = \sin 43.6 \times \frac{2.4}{\sin 67.5} = \mathbf{1.79\ cm}\ (2\ \mathrm{d.p.})$$

Cosine Rule

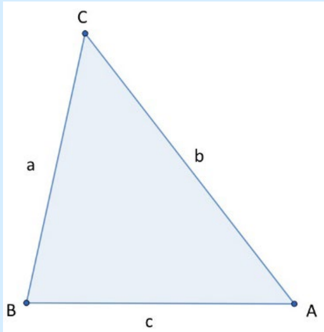
In addition to the sine rule, the cosine rule also allows us to work out angles and lengths of a triangle.

We use the cosine rule when we have been given slightly different information, usually the **length of two sides** and the **size of the angle between these sides**. We cannot use the sine rule for this because we do not know one angle and its opposite side.
The cosine rule is:

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc\cos A}$$

Example: In the following triangle, side b is 7 cm, side c is 5.5 cm and angle A is $52.1°$.

Calculate the length of side a using the cosine rule.



Substitute the values into the cosine rule formula and solve to find a:

$$a^2 = 7^2 + 5.5^2 - 2 \times 7 \times 5.5 \times \cos 52.1$$

$$a^2 = 31.95003959 \ldots.$$

$$a = \sqrt{31.95003959} = 5.65\ (2\ \mathrm{d.p.})$$

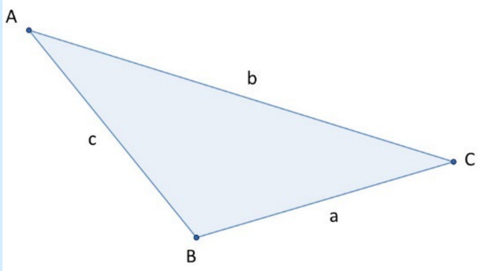
We may be **given the lengths** of all three sides of a triangle and asked to **find an angle**. To do this, we need to **rearrange** the cosine equation so that we have $\cos A$ on one side:

$$\boldsymbol{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}$$

This equation can be used to find an unknown angle, as long as we are given all the lengths of the triangle.

Example: In the following triangle, side a is 5 cm, side b is 8.2 cm and side c is 4.9 cm.

What is the size of angle A?



1. Choose which formula to use to calculate angle A from the given values.

   Since we are given all the length measurements of a, b and c, we use the following rearranged form of the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

2. Substitute the given values into the cosine rule formula to find angle A.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{8.2^2 + 4.9^2 - 5^2}{2 \times 8.2 \times 4.9}$$

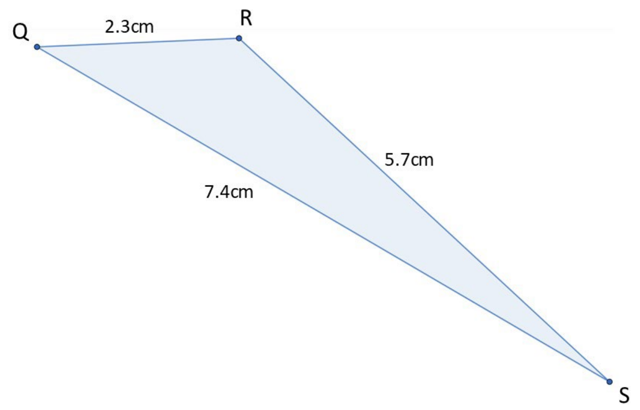
$$\cos A = 0.8244151319 \ldots$$

   *Now use the **inverse cos** button ($\cos^{-1}$) to work out the angle:*

$$\cos^{-1}(0.8244151319 \ldots) = \mathbf{34.47^\circ}\ (2\text{ d.p.})$$

Example: In a triangle QRS, side QR has a length of 2.3 cm, RS has a length of 5.7 cm and QS has a length of 7.4 cm. Calculate the side of the angle between sides QR and RS.

1. Sketch the triangle to help understand the question.



We are trying to find the angle at R, so let's call this A for now. If we say Q is now B and S is now C, the opposite sides of these angles can be b and c, which fits into the rearranged cosine equation:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

2. *Substitute the known values into the formula and find the required angle.*

$$\cos A = \frac{5.7^2 + 2.3^2 - 7.4^2}{2 \times 5.7 \times 2.3}$$

$$\cos A = -0.647597254$$

Taking the inverse of this ($\cos^{-1}$) *gives us*

$$\boldsymbol{A = 130.36°}\ (2\ \mathrm{d.p.})$$

This angle makes sense because the angle at point R is clearly obtuse.

Deciding whether to use the sine or the cosine rule depends on the information we have available to us.

- If we know **two angles** and **one opposite side**, or **two sides** and one of their **opposite angles**, then we use the **sine rule**.
- If we know **all three side lengths**, or two lengths and the size of the angle **between them**, then we use the **cosine rule**.

Area of a Triangle

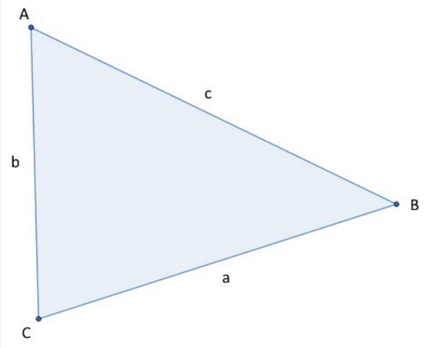
The formula we use to calculate the **area** of a triangle is:

$$\boldsymbol{Area = \frac{1}{2}ab\sin C}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ are two lengths of the triangle and $\boldsymbol{C}$ is the angle between them.

Example: In the following triangle, side a is 4.1 cm, side b is 3.2 cm and angle C is $73.7°$.

Calculate the area of the triangle.



1. Recall the formula for the area of the triangle.

$$Area = \frac{1}{2}ab\sin C$$

2. Substitute in the given values and compute the area.

$$Area = \frac{1}{2} \times 4.1 \times 3.2 \times \sin 73.7$$

$$Area = \mathbf{6.30\ cm^2}\ (2\ \mathrm{d.p})$$

Don't forget to use the **correct units** for area!

Example: Find the area of a triangle when the angle between two sides, each measuring 4.5 cm and 3.7 cm, measures $46°$.

1. Recall the formula for the area of the triangle.

$$Area = \frac{1}{2}ab\sin C$$

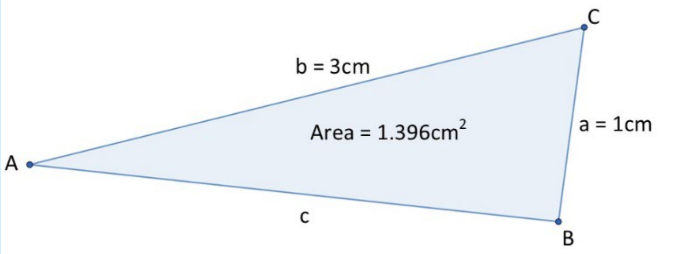
2. Substitute in the given values and compute the area.

$$Area = \frac{1}{2} \times 4.5 \times 3.7 \times \sin 46$$

$$Area = \mathbf{5.99\ cm^2}\ (2\ \mathrm{d.p.})$$

We can **rearrange the formula for area** to find a different value.

Example: Calculate angle ACB in the following triangle.



1. Rearrange the formula for the area of a triangle to give an expression where the angle is part of the subject of the equation.

$$Area = \frac{1}{2}ab\sin C \quad \rightarrow \quad \sin C = \frac{2 \times Area}{ab}$$

2. Substitute in the given values to find angle ACB.

$$\sin C = \frac{2 \times 1.396}{1 \times 3} = 0.9306666667 \ldots$$

Use the inverse sine button ($\sin^{-1}$) to work out the angle.

$$\sin^{-1} 0.9306666667 = \mathbf{68.5}° \text{ (2 d.p)}$$

Example: The area of a triangle is 9.35 cm^2. One side measures 4.6 cm and another measures 7.2 cm. Calculate the size of the angle at which these two sides meet.

1. Rearrange the formula for the area of a triangle to give an expression where the angle is part of the subject of the equation.

$$\sin C = \frac{2 \times Area}{ab}$$

3. Substitute in the given values to find angle ACB.

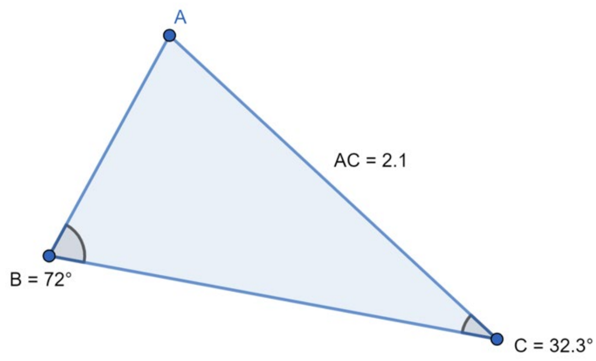
$$\sin C = \frac{2 \times 9.35}{4.6 \times 7.2} = 0.5646135266 \ldots$$

Use the inverse sine button ($\sin^{-1}$) to work out the angle.

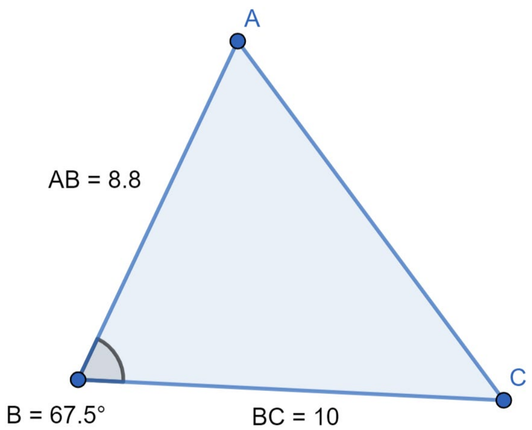
$$\sin^{-1} 0.5646135266 = \mathbf{34.38}° \text{ (2 d.p)}$$

Sine and Cosine Rules and Area of a Triangle – Practice Questions

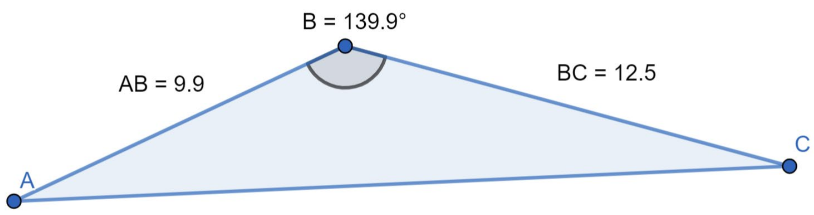
1. Calculate the length of AB in the following triangle:



2. Calculate the length of AC in the following triangle:



3. Calculate the area of the following triangle:



Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.