GCSE Maths – Algebra

Equations of a Circle and its Tangent (Higher Only)

Notes

WORKSHEET

Equation of a Circle

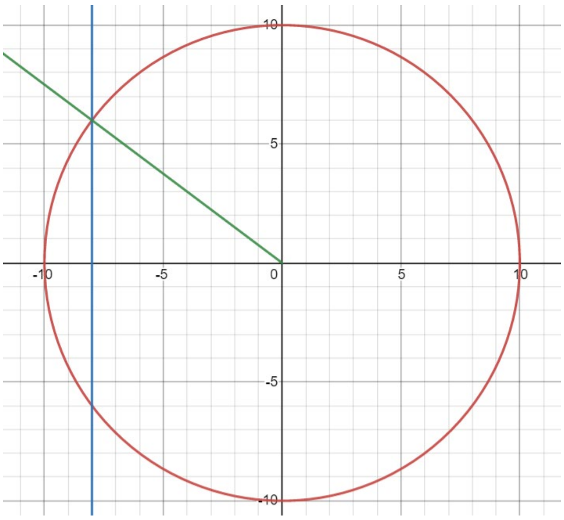
The basic equation of a circle centred at the origin $(0,0)$ is

$$\boldsymbol{x^2 + y^2 = r^2},$$

where *r* is the radius of the circle.

This equation bears resemblance to **Pythagoras' theorem** for a reason – a right-angled triangle can be made within a circle with the **hypotenuse as the radius** (shown as the green line below).

Look at the following graph:



The equation of the red circle is $\boldsymbol{x^2 + y^2 = 100}$. This means that for any value of x and any value of y such that $x^2 + y^2 = 100$, then that point (x, y) will **lie on the circle**.

> For example, let's consider the point $(-8, -6)$. We know that this point will lie on the circle, because $(-8)^2 + (-6)^2 = 100$.

On the graph, the blue line is $x = -8$. This line forms one **side of the right-angled triangle** within the circle, with the x-axis being the other side and the green line being the **hypotenuse**.

Using Pythagoras' Theorem, we can calculate the length of the green line from the origin $(0, 0)$ to the point where it intersects the circle.

> As $a^2 + b^2 = c^2$, we can substitute in the lengths of each side of the triangle:

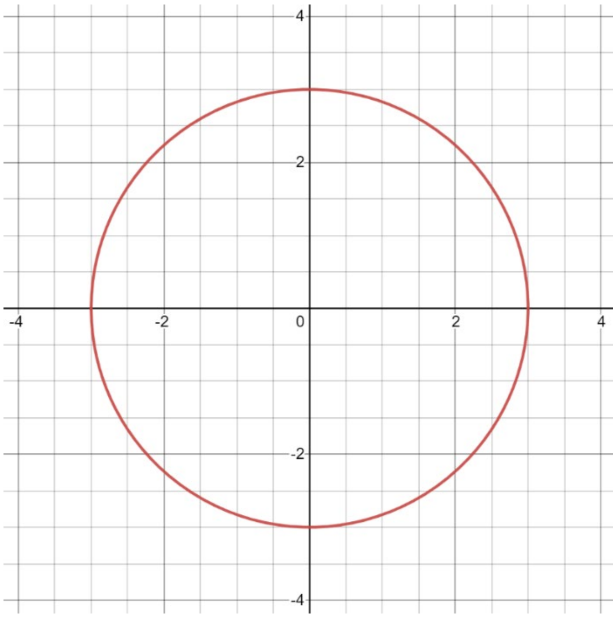
$$(-8)^2 + (6)^2 = 100$$
$$\sqrt{100} = 10$$

Therefore, the length of the green line from the origin to $(-8, 6)$ is $\mathbf{10}$. This length is the **radius** of the circle. This illustrates why the equation of a circle is $\boldsymbol{x^2 + y^2 = r^2}$.

Finding the equation of a circle

To work out the equation of the circle, we need to calculate its radius.

For example, consider the following graph.



Let's work out the radius of this circle:

An easy way to do this is to substitute in either $x = 0$ or $y = 0$ and read from the graph. When $y = 0$, we can see that the circle crosses the x-axis at 3 and -3.

We know two points of the circle now: $(3, 0)$ and $(-3, 0)$.
One of these points can be substituted into the basic equation of a circle to find r:

$$x^2 + y^2 = r^2 \quad \rightarrow \quad 3^2 + 0^2 = r^2 \quad \rightarrow \quad r^2 = 9 \quad \rightarrow \quad \boldsymbol{r = 3}$$

This tells us the radius is 3. Therefore, the equation of this circle is $\boldsymbol{x^2 + y^2 = 9}$.

Example: A circle has the equation $x^2 + y^2 = 60$. Calculate the radius of this circle.

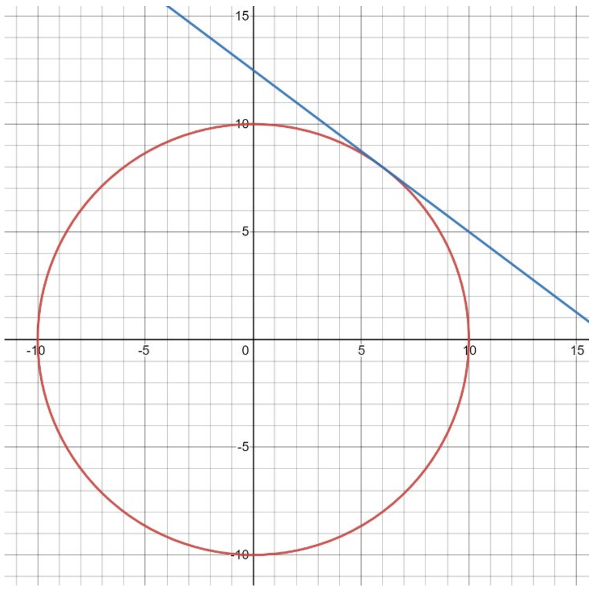
Using the equation of a circle, $\boldsymbol{x^2 + y^2 = r^2}$*, we can deduce that* $r^2 = 60$.

Taking the square root of this will give us the radius:

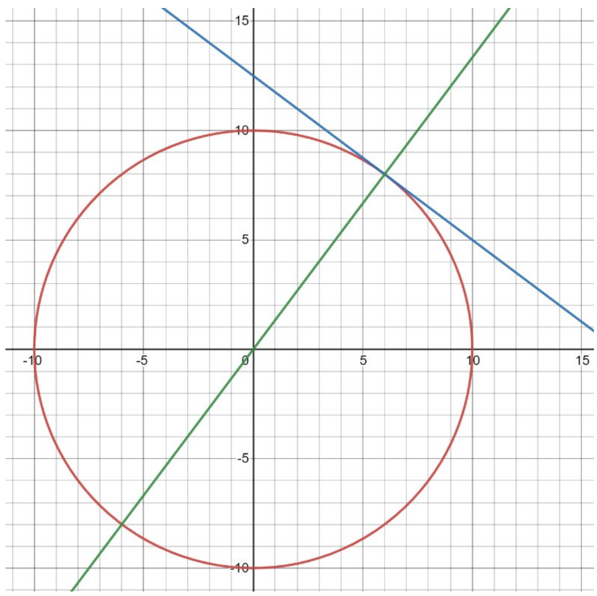
$$r = \sqrt{60} = \mathbf{7.746}\ (3\ d.p.)$$

Tangents to the circle

A tangent is a **straight line** that just **touches** the circle.
Consider the following graph, which shows the circle with equation $x^2 + y^2 = 100$:



The blue line is a tangent to the circle at the point $(6, 8)$. Crucially, the tangent is **perpendicular to the radius** at this point. This means it meets the radius at a right angle:



The green line shows the **radius** from the origin to the point $(6, 8)$. As we can see, the tangent and the radius are **perpendicular**. In terms of their gradients, this means that the gradient of the tangent is the **negative reciprocal** of the radius.

Finding the equation of a tangent to a circle

We can use the radius to calculate the equation of the tangent at a point.

For the circle shown on the previous graph, we can find the tangent to the point at $(6,8)$ as follows:

1. Work out the gradient of the radius which passes through the required point $(6,8)$.

   To calculate the gradient, work out the ***difference*** *in the* ***y coordinates*** *divided by the* ***difference*** *in* ***x coordinates*** *of the points it passes through. We know it passes through* $(0,0)$ *and* $(6,8)$*:*

   $$Gradient = \frac{8-0}{6-0} = \frac{8}{6} = \frac{4}{3}$$

2. The gradient of the tangent is the **negative reciprocal** of the gradient of the radius. Therefore, we **flip the fraction** (numerator and denominator) and make it negative (or make it positive if the radius gradient is already negative).

   This means that the gradient of the tangent is $-\frac{3}{4}$.

3. We find the equation of the tangent by working out the **y-intercept**, or *c* in the equation $y = mx + c$.

   To calculate c*,* ***substitute*** *in the values for a coordinate which we know that the tangent passes through, e.g.* $(6, 8)$.

   $$8 = -\frac{3}{4} \times 6 + c$$
   $$c = 12.5$$

   Now that we know the y-intercept, we can write out the full equation of the tangent:

   $$\boldsymbol{y = -\frac{3}{4} + 12.5}$$

Example: A circle has the equation $x^2 + y^2 = 50$. Work out the equation of the tangent that touches the circle at $(-1, 7)$.

1. Work out the gradient of the radius which passes through $(-1,7)$.

   The radius passes through points (0, 0) and (-1, 7).

   $$Gradient = \frac{7-0}{-1-0} = \boldsymbol{-7}$$

2. Find the gradient of the tangent by taking the negative reciprocal of the radius gradient.

   $$Gradient\ of\ tangent = \frac{1}{7}.$$

3. Find the y-intercept of the tangent.

   Substitute in the values for x and y that we know the tangent passes through:

   $$7 = \frac{1}{7} \times -1 + c \quad \rightarrow \quad c = \frac{50}{7}$$

   The equation of the tangent is $\boldsymbol{y = \frac{1}{7}x + \frac{50}{7}}$.

Example: A circle has the equation $x^2 + y^2 = 73$. A tangent touches the circle at $(8, 3)$. Find the equation of this tangent.

1. Find the gradient of the radius which passes through $(8,3)$.

   To do this, work out the difference in y-coordinates and x-coordinates between the origin $(0,0)$ *and the point* $(8,3)$.

$$Gradient\ of\ radius = \frac{3-0}{8-0} = \frac{3}{8}$$

2. Take the negative reciprocal of the gradient of the radius to find the gradient of the tangent.

$$Gradient\ of\ tangent\ = -\frac{8}{3}.$$

3. Substitute in the values of x and y at the point $(8, 3)$ to work out the y-intercept of the tangent.

$$3 = -\frac{8}{3} \times 8 + c$$

$$3 = -\frac{64}{3} + c$$

$$c = \frac{73}{3}$$

Therefore, the equation of the tangent is:

$$\boldsymbol{y = -\frac{8}{3}x + \frac{73}{3}}$$

Equations of a Circle and its Tangent – Practice Questions

1. What is the equation of a circle that has a radius of 9?

2. What is the radius of a circle that has the equation $x^2 + y^2 = 56$?

3. A circle has the equation $x^2 + y^2 = 85$. Find the equation of the tangent that touches the circle at $(-2, 9)$.

4. A circle has equation $\frac{x^2+y^2}{2} = 40$. Find the equation of the tangent that touches the circle at $(8, 4)$.

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.