GCSE Maths – Algebra

Roots, Intercepts and Turning Points

Notes

WORKSHEET

Roots

A root is a **solution** to a **quadratic equation** when it is set **equal to zero**. This means roots are the points at which a quadratic $ax^2 + bx + c$ crosses the x axis.

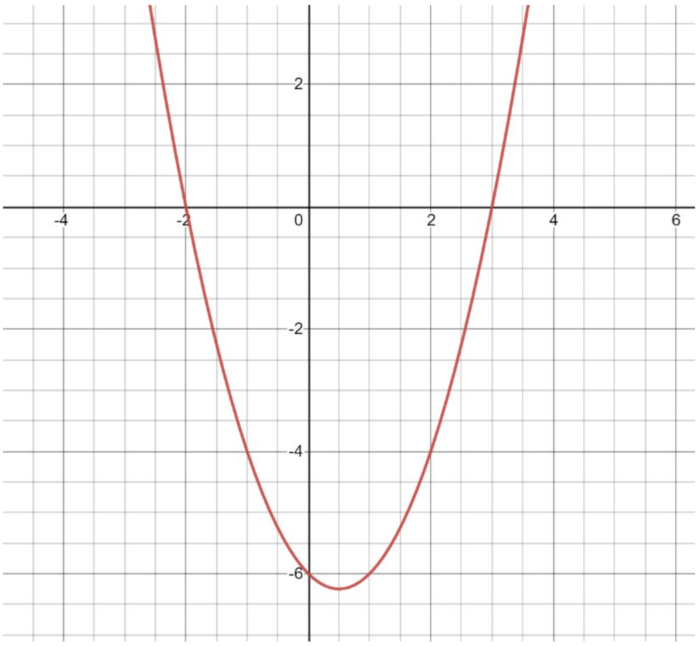
For example, take the equation $x^2 + 3x + 2 = 0$.
By **factorising**, we can write this as

$$x^2 + 3x + 2 = (x + 1)(x + 2) = 0.$$

The solutions of this equation are $x = -1$ and $x = -2$, because **substituting** in these values for x will equal 0. So, the roots of the quadratic $x^2 + 3x + 2$ are $x = -1$ and $x = -2$.

A quadratic equation corresponds to a **curve** on a graph, and we can **identify the roots** to a quadratic equation by looking at the graph.

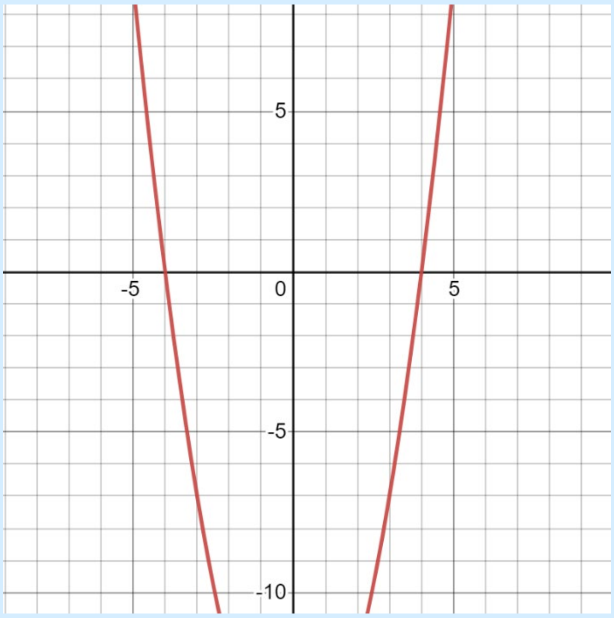
For example, consider the graph for the quadratic equation $y = x^2 + x - 6$:



We could **factorise** this equation to find the roots, or we could use the graph.
To find the roots graphically, we are looking for the **values of x that satisfies $y = 0$**. To put this another way, we are looking at where the curve **touches or crosses the x-axis**, as this is where $y = 0$.

This curve **crosses the x-axis** at $x = -2$ and $x = 3$. These are the values of the **roots**.
To **check** this, we could factorise the quadratic equation $x^2 + x - 6 = 0$, which is $(x - 3)(x + 2) = 0$. This gives us the same solutions for the roots.

Example: Use the graph to find the roots of the equation $x^2 - 16 = 0$.



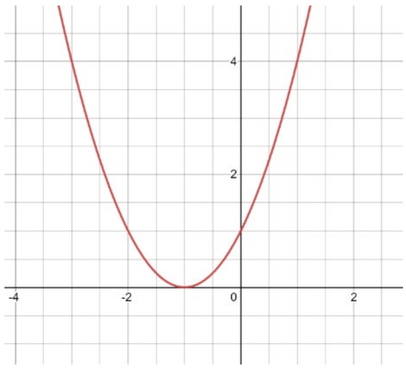
To find the roots graphically, we are looking for where the curve touches or crosses the x-axis, because this is where $y = 0$.

The curve crosses at $\boldsymbol{x = -4}$ and $\boldsymbol{x = 4}$. These solutions are the roots to the equation.

- A quadratic curve may touch the x-axis only once, meaning there is **only one root**.

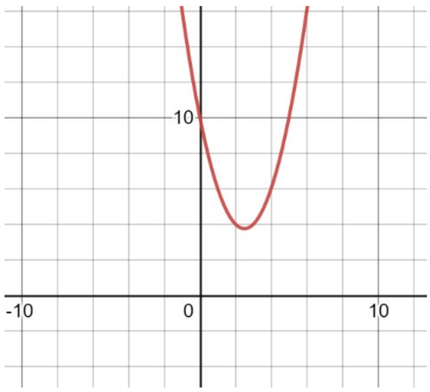
  For example, the following graph shows the curve for $y = x^2 + 2x + 1$:

  The curve only touches the x-axis **once**, at $x = -1$. This is because it is the only root to the equation. If we were to factorise the equation to $(x + 1)(x + 1) = 0$, we can see that $x = -1$ is the only solution.



- Quadratic curves may **not touch or cross the x-axis** at all. The following graph shows $y = x^2 - 5x + 10$:

  The curve does not touch the x-axis at all, meaning there are **no roots** to this equation. Similarly, if we try to factorise this equation, or use the quadratic formula, we cannot find a solution to this equation.



Intercepts

The intercept is the **place where the curve crosses the y-axis**. At this point, $x = 0$. To find the y-intercept, we **substitute $x = 0$** into our equation to find the value of y.

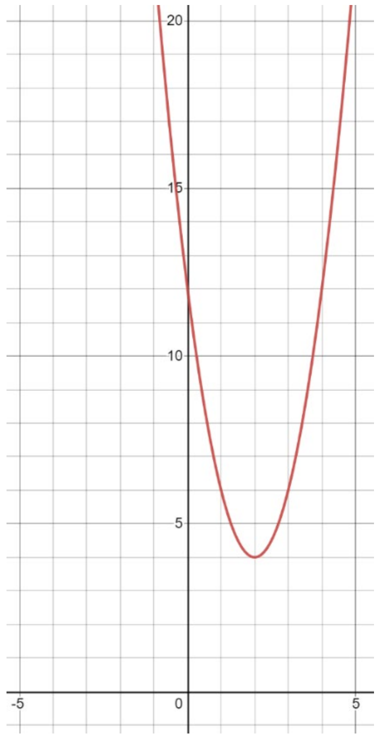
For example, consider the equation $y = x^2 - 7x + 8$.
To find where this curve crosses the y-axis, substitute in $x = 0$:

$$y = 0^2 - (7 \times 0) + 8 = \mathbf{8}$$

Now that we know the value of y, we can say that the curve crosses the y-axis at $(0, 8)$. This is where it **intercepts the y-axis**.

We can also use the **graph** to find the y-intercept.

The graph shows the equation $y = 2x^2 - 8x + 12$.



The fact that this curve doesn't have any roots doesn't matter – we are only looking to find the **y-intercept**.

Using the graph, we can see that the curve crosses the y-axis at $y = 12$.

To check this, we should substitute in $x = 0$ into the equation:

$$y = 2 \times 0^2 - (8 \times 0) + 12$$
$$\boldsymbol{y = 12}$$

Example: Find the y-intercept for the curve $y = x^2 + 9x - 14$.

The place where the curve crosses the y-axis is when $x = 0$.
Therefore, substitute $x = 0$ into the equation:

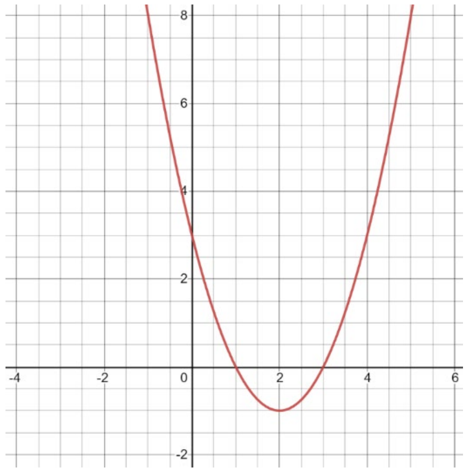
$$y = 0^2 + (9 \times 0) - 14$$
$$\boldsymbol{y = -14}$$

The curve crosses the y-axis at $(0, -14)$ and the y-intercept is at $y = -14$.

Turning Points

A turning point is the place on a curve where it **changes direction**. At the turning point, the gradient is 0.
For example, look at the following graph, which shows the equation $y = x^2 - 4x + 3$:

Using the graph, we can locate the turning point at $(2, -1)$, as this is where the curve changes direction.



Completing the Square (Higher Only)

To find the turning point using algebra, we need to **complete the square**.
To work through this, consider the equation $x^2 - 8x + 12 = 0$.

1. Using the general equation $ax^2 + bx + c = 0$, identify which values represent a, b and c in the given quadratic equation.

   For $x^2 - 8x + 12 = 0$, we have $a = 1$, $b = -8$ and $c = 12$.
2. Complete the square for the quadratic by using the formula $\mathbf{(x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2 + c = 0.}$

$$(x-4)^2 - (-4)^2 + 12 = 0$$
$$(x-4)^2 - 4 = 0$$

   This is a graph on a curve, so it can be written as $y = (x-4)^2 - 4$.

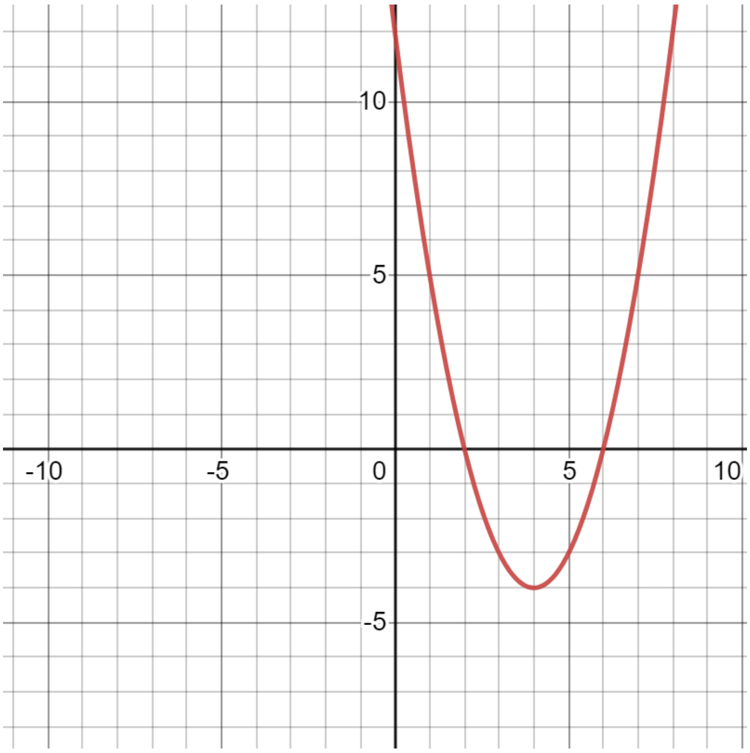
3. We now need to find the value of x that gives us the **lowest possible value of *y***, because this is the **lowest point of the curve** and is therefore the **turning point**. Since $(x - 4)$ is squared, it will always give us a **positive number**, so we need to make this **term equal to 0** to give us the lowest possible value for y.
   Therefore, $x = 4$ at the turning point.

4. Now that we know the value of *x* at the turning point, **substitute** it in to find the value of *y*:

$$y = (4-4)^2 - 4$$

   This gives us $y = -4$.
   The coordinates for the turning point are $(\mathbf{4}, -\mathbf{4})$.

Here is the graph for this quadratic curve:



We can see that the turning point is indeed at $(4, -4)$.

Example: Find the turning point for the curve described by the equation $y = x^2 + 10x - 11$

1. By completing the square, write the equation in the form $y = (x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2 + c$.

$$y = (x + 5)^2 - (5)^2 - 11$$

This can be simplified to

$$y = (x + 5)^2 - 36.$$

2. Identify the value of x that gives the lowest possible value of y.

This means we need to find the value of x that makes $(x + 5)^2 = 0$

$$x = -5$$

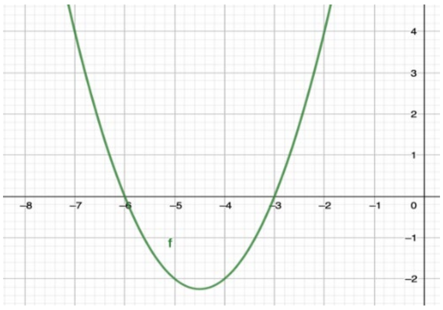
3. Calculate the value of y when $x = -5$.

$$y = (0 + 5)^2 - 36 = -36.$$

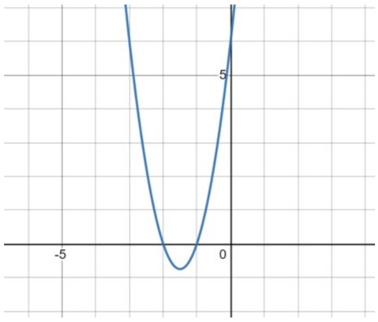
The coordinates for the turning point are $(\mathbf{-5, -36})$.

Roots, Intercepts and Turning Points – Practice Questions

1. Find the number of roots and their values for the equation $f(x) = x^2 + 9x + 18$

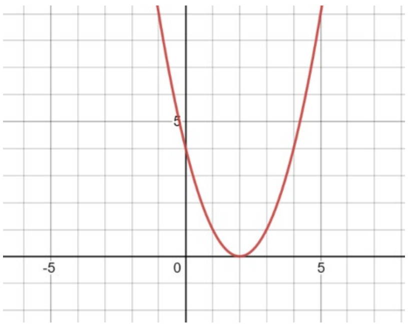


2. Use the following graph to find the y-intercept of the curve given by the equation $f(x) = 9x^2 - 8x + 6$.



3. Calculate the y-intercept of the curve given by the equation $f(x) = 7x^2 + 8x - 2$.

4. Use the graph to find the turning points for the curve $y = x^2 - 4x + 4$.



5. By completing the square, find the turning points of the curve $y = x^2 - 6x + 2$.

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.