Q1				
(i)	$L = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(W_1-\mu)^2}{2\sigma_1^2}} \cdot \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(W_2-\mu)^2}{2\sigma_2^2}}$	M1 M1 A1	Product form. Two Normal terms. Fully correct.	
	$\ln L = \text{const} - \frac{1}{2\sigma_1^2}(W_1-\mu)^2 - \frac{1}{2\sigma_2^2}(W_2-\mu)^2$	M1 A1		
	$\frac{d\ln L}{d\mu} = \frac{2}{2\sigma_1^2}(W_1-\mu) + \frac{2}{2\sigma_2^2}(W_2-\mu)$	M1 A1	Differentiate w.r.t. μ.	
	$= 0 \Rightarrow \sigma_2^2 W_1 - \sigma_2^2\mu + \sigma_1^2 W_2 - \sigma_1^2\mu = 0$	A1		
	$\Rightarrow \hat{\mu} = \frac{\sigma_2^2 W_1 + \sigma_1^2 W_2}{\sigma_1^2 + \sigma_2^2}$	A1	BEWARE PRINTED ANSWER.	
	Check this is a maximum.	M1		
	E.g. $\frac{d^2\ln L}{d\mu^2} = -\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} < 0$	A1		11
(ii)	$E(\hat{\mu}) = \frac{\sigma_2^2\mu + \sigma_1^2\mu}{\sigma_1^2 + \sigma_2^2} = \mu$	M1		
	$\therefore$ unbiased.	A1		2
(iii)	$\text{Var}(\hat{\mu}) = \left(\frac{1}{\sigma_1^2 + \sigma_1^2}\right)^2 \cdot (\sigma_2^4\sigma_1^2 + \sigma_1^4\sigma_2^2)$	B1 B1	First factor. Second factor.	
	$= \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$		Simplification not required at this point.	2
(iv)	$T = \frac{1}{2}(W_1 + W_2)$			
	$\text{Var}(T) = \frac{1}{4}(\sigma_1^2 + \sigma_2^2)$	B1		
	Relative efficiency (y) $= \frac{\text{Var}(\hat{\mu})}{\text{Var}(T)}$	M1 M1	Any attempt to compare variances. If correct.	
	$= \frac{\sigma_2^4\sigma_1^2 + \sigma_1^4\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} \cdot \frac{4}{\sigma_1^2 + \sigma_2^2}$	A1		
	$= \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2}$	A1	BEWARE PRINTED ANSWER.	5
(v)	E.g. consider $\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 = (\sigma_1 - \sigma_2)^2 \geq 0$	M1		
	$\therefore$ Denominator $\geq$ numerator, $\therefore$ fraction $\leq$ 1	E1		
	[Both $\hat{\mu}$ and T are unbiased,] $\hat{\mu}$ has smaller variance than T and is therefore better.	E1 E1		4
				24

Q2	$f(x) = \dfrac{\lambda^{k+1} x^k e^{-\lambda x}}{k!}$, $[x > 0 \quad (\lambda > 0,\ k \text{ integer} \geq 0)]$ Given: $\int_0^\infty u^m e^{-u} du = m!$			
(i)	$M_X(\theta) = E[e^{\theta x}]$ $= \int_0^\infty \dfrac{\lambda^{k+1}}{k!} x^k e^{-(\lambda-\theta)x} dx$ Put $(\lambda - \theta)x = u$ $= \dfrac{\lambda^{k+1}}{k!(\lambda-\theta)^{k+1}} \int_0^\infty u^k e^{-u} du$ $= \left(\dfrac{\lambda}{\lambda-\theta}\right)^{k+1}$	M1 M1 M1 A1 A1 A1 A1	For obtaining this expression after substitution. Take out constants. (Dep on subst.) Apply "given": integral = k! (Dep on subst.) BEWARE PRINTED ANSWER.	7
(ii)	$Y = X_1 + X_2 + \ldots + X_n$ By convolution theorem:- mgf of Y is $\{M_X(\theta)\}^n$ i.e. $\left(\dfrac{\lambda}{\lambda-\theta}\right)^{nk+n}$ $\mu = M'(0)$ $M'(\theta) = \lambda^{nk+n}(-nk-n)(\lambda-\theta)^{-nk-n-1}(-1)$ $\therefore \mu = \dfrac{nk+n}{\lambda}$ $\sigma^2 = M''(0) - \mu^2$ $M''(\theta) = (nk+n)\lambda^{nk+n}(-nk-n-1)(\lambda-\theta)^{-nk-n-2}(-1)$ $\therefore M''(0) = (nk+n)(nk+n+1)/\lambda^2$ $\therefore \sigma^2 = \dfrac{(nk+n)(nk+n+1)}{\lambda^2} - \dfrac{(nk+n)^2}{\lambda^2}$ $= \dfrac{nk+n}{\lambda^2}$	B1 M1 A1 A1 M1 A1 M1 A1		8
(iii)	[Note that $M_Y(t)$ is of the same functional form as $M_X(t)$ with $k + 1$ replaced by $nk + n$, i.e. k replaced by $nk + n - 1$. This must also be true of the pdf.] Pdf of Y is $\dfrac{\lambda^{nk+n}}{(nk+n-1)!} \times y^{nk+n-1} \times e^{-\lambda y}$ [for $y > 0$]	B1 B1 B1	One mark for each factor of the expression. Mark for third factor shown here depends on at least one of the other two earned.	3
(iv)	$\lambda = 1$, $k = 2$, $n = 5$, Exact $P(Y > 10) = 0{\cdot}9165$ Use of N(15, 15)	M1 M1	Mean. ft (ii). Variance. ft (ii).	

	P(this > 10) = $P\left(N(0,1) > \frac{10-15}{\sqrt{15}} = -1\cdot 291\right)$ = 0·9017 Reasonably good agreement – CLT working for only small *n*.	A1 A1 E2	c.a.o. c.a.o. (E1, E1) [Or other sensible comments.]	6
				24

Q3				
(i)	$\bar{x} = 36.48$ $s = 9.6307$ $s^2 = 92.7507$ $\bar{y} = 45.5$ $s = 14.8129$ $s^2 = 219.4218$	B1	If all correct. [No marks for use of s_n which are 9.1365 and 14.1823 respectively.]	
	Assumptions: Normality of <u>both</u> populations equal variances	B1 B1		
	$H_0 : \mu_A = \mu_B$ $H_1 : \mu_A \neq \mu_B$	B1	Do <u>NOT</u> accept $\bar{X} = \bar{Y}$ or similar.	
	Where μ_A, μ_B are the population means.	B1		
	Pooled $s^2 = \frac{9 \times 92.7507 + 11 \times 219.4218}{20}$ $= \frac{834.756 + 24136.64}{20} = 162.4198$	B1	$= (12.7444)^2$	
	Test statistic is $\frac{36.48 - 45.5}{\sqrt{162.4198}\sqrt{\frac{1}{10} + \frac{1}{12}}}$	M1		
	$= \frac{-9.02}{5.4568} = -1.653$	A1		
	Refer to t_{20}.	M1	No ft from here if wrong.	
	Double tailed 5% point is 2·086.	A1	No ft from here if wrong.	
	Not significant.	A1	ft only c's test statistic.	
	No evidence that population mean times differ.	A1	ft only c's test statistic.	12
(ii)	Assumption: Normality of underlying population of <u>differences</u>.	B1		
	$H_0 : \mu_D = 0$ $H_1 : \mu_D > 0$	B1	Do <u>NOT</u> accept $\bar{D} = 0$ or similar.	
	Where μ_D is the population mean of "before – after" differences.	B1	The "<u>direction</u>" of D must be CLEAR. Allow $\mu_A = \mu_B$ etc.	
	Differences are 6.4, 4.4, 3.9, -1.0, 5.6, 8.8, -1.8, 12.1	M1		
	($\bar{x} = 4.8$ $s = 4.6393$)		[A1 can be awarded here if NOT awarded in part (i)]. Use of s_n (=4.3396) is <u>NOT</u> acceptable, even in a denominator of $\frac{s_n}{\sqrt{n-1}}$	
	Test statistic is $\frac{4.8 - 0}{4.6393/\sqrt{8}}$	M1		
	=2.92(64)	A1		
	Refer to t_7.	M1	No ft from here if wrong.	
	Single tailed 5% point is 1.895.	A1	No ft from here if wrong.	
	Significant.	A1	ft only c's test statistic.	
	Seems mean is lowered.	A1	ft only c's test statistic.	10
(iii)	The paired comparison in part (ii) eliminates the variability between workers.	E2	(E1, E1)	2
				24

<table>
<tr><td>Q4</td><td></td><td></td><td></td><td></td></tr>
<tr><td>(i)</td><td>Latin square.

Layout such as:

Locations
<table>
<tr><th></th><th></th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td></td><td>I</td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td></tr>
<tr><td>Surf</td><td>II</td><td>B</td><td>C</td><td>D</td><td>E</td><td>A</td></tr>
<tr><td>-aces</td><td>III</td><td>C</td><td>D</td><td>E</td><td>A</td><td>B</td></tr>
<tr><td></td><td>IV</td><td>D</td><td>E</td><td>A</td><td>B</td><td>C</td></tr>
<tr><td></td><td>V</td><td>E</td><td>A</td><td>B</td><td>C</td><td>D</td></tr>
</table></td><td>B1

B1

B1</td><td>(letters = paints)
Correct rows and columns.

A correct arrangement of letters.
SC. For a description instead of an example allow max 1 out of 2.</td><td>3</td></tr>
<tr><td>(ii)</td><td>$X_{ij} = \mu + \alpha_i + e_{ij}$

μ = population
grand mean for whole experiment.

α_i = population
mean amount by which the i^{th} treatment differs from μ.

e_{ij} are experimental errors
~ ind
$N(0, \sigma^2)$.</td><td>B1

B1
B1

B1

B1

B1
B1
B1
B1</td><td>

Allow "uncorrelated".
Mean.
Variance.</td><td>9</td></tr>
<tr><td>(iii)</td><td>Totals are: 322, 351, 307, 355, 291
(each from sample of size 5)
Grand total: 1626
"Correction factor" CF = $\frac{1626^2}{25} = 105755.04$

Total SS = 106838 – CF = 1082.96
Between paints SS = $\frac{322^2}{5} + ... + \frac{291^2}{5} - CF$
= 106368 – CF =612.96
Residual SS (by subtraction) = 1082.96 – 612.96
= 470.00

<table>
<tr><th>Source of variation</th><th>SS</th><th>df</th><th>MS</th></tr>
<tr><td>Between paints</td><td>612.96</td><td>4</td><td>153.24</td></tr>
<tr><td>Residual</td><td>470.00</td><td>20</td><td>23.5</td></tr>
<tr><td>Total</td><td>1082.96</td><td>24</td><td></td></tr>
</table>
MS ratio = $\frac{153.24}{23.5} = 6.52$</td><td>M1
M1

A1

B1
B1
M1

M1
A1</td><td>
For correct methods for any two SS.

If each calculated SS is correct.

Degrees of freedom "between paints".
Degrees of freedom "residual".
MS column.

Independent of previous M1.
Dep only on this M1.</td><td></td></tr>
</table>

	Refer to $F_{4,\ 20}$	M1	No ft if wrong. But allow ft of wrong d.o.f. above.	
	Upper 5% point is 2.87	A1	No ft if wrong.	
	Significant.	A1	ft only c's test statistic and d.o.f.'s.	
	Seems performances of paints are not all the same.	A1	ft only c's test statistic and d.o.f.'s.	12
				24

1) (i)	$f(x) = \frac{1}{\theta} \quad 0 \le x \le \theta$ $E[X] = \frac{\theta}{2}$ $E[2\overline{X}] = 2E[\overline{X}] = 2E[X]$ $= \theta$ $\therefore$ unbiased	 B1 M1 A1 E1	 Write-down, or by symmetry, or by integration.	 4
(ii)	$\sum x = 2.3 \quad \therefore \ \overline{x} = \frac{2.3}{5} = 0.46 \quad \therefore 2\overline{x} = 0.92$ But we know $\theta \ge 1$ $\therefore$ estimator can give nonsense answers, i.e. essentially useless	B1 E1 E2	 (E1, E1)	 4
(iii)	$Y = \max\{X_i\}$, $g(y) = \frac{ny^{n-1}}{\theta^n} \quad 0 \le y \le \theta$ MSE $(kY) = E[(kY-\theta)^2] =$ $E[k^2Y^2 - 2k\theta Y + \theta^2] =$ $k^2E[Y^2] - 2k\theta E[Y] + \theta^2$ $\frac{d\text{MSE}}{dk} =$ $2kE[Y^2] - 2\theta E[Y] = 0$ for $k = \frac{\theta E[Y]}{E[Y^2]}$ $\frac{d^2\text{MSE}}{dk^2} = 2E[Y^2] > 0 \quad \therefore$ this is a minimum $E[Y] = \int_0^\theta \frac{ny^n}{\theta^n} dy = \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1}$ $E[Y^2] = \int_0^\theta \frac{ny^{n+1}}{\theta^n} dy = \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$ $\therefore$ minimising $k = \theta \frac{n\theta}{n+1} \frac{n+2}{n\theta^2} = \frac{n+2}{n+1}$	 M1 1 M1 M1 A1 M1 M1 A1 M1 A1 M1 A1	 BEWARE PRINTED ANSWER	 12
(iv)	With this k, kY is always greater than the sample maximum So it does not suffer from the disadvantage in part (ii)	E2 E2	(E1 E1) (E1 E1)	 4

2(i)	$G(t)=E[t^X]=\sum_{x=0}^{n}\binom{n}{x}(pt)^x(1-p)^{n-x}$ $=[(1-p)+pt]^n$ $=(q+pt)^n$	M1 2 1	Available as B2 for write-down or as 1+1 for algebra	4
(ii)	$\mu=G'(1)\quad G'(t)=np(q+pt)^{n-1}$ $G'(1)=np\times 1=np$ $\sigma^2=G''(1)+\mu-\mu^2$ $G''(t)=n(n-1)p^2(q+pt)^{n-2}$ $G''(1)=n(n-1)p^2$ $\therefore \sigma^2=n^2p^2-np^2+np-n^2p^2$ $=-np^2+np=npq$	1 1 1 1 M1 1		6
(iii)	$Z=\dfrac{X-\mu}{\sigma}$ Mean 0, Variance 1	B1	For <u>BOTH</u>	1
(iv)	$M(\theta)=G(e^\theta)=(q+pe^\theta)^n$ $Z=aX+b$ with: $a=\dfrac{1}{\sigma}=\dfrac{1}{\sqrt{npq}}$ and $b=-\dfrac{\mu}{\sigma}=-\sqrt{\dfrac{np}{q}}$ $M_Z(\theta)=e^{b\theta}M_X(a\theta)$ $\therefore M_Z(\theta)=e^{-\sqrt{\frac{np}{q}}\theta}\left(q+pe^{\frac{1}{\sqrt{npq}}\theta}\right)^n=$ $\left(qe^{-\frac{p\theta}{\sqrt{npq}}}+pe^{\frac{1-p}{\sqrt{npq}}\theta}\right)^n$	1 M1 1 1 1	BEWARE PRINTED ANSWER	5
(v)	$M_Z(\theta)=(q-\dfrac{qp\theta}{\sqrt{npq}}+\dfrac{qp^2\theta^2}{2npq}+$ terms in $n^{-3/2}$, n^{-2}, ………….. + $p+\dfrac{pq\theta}{\sqrt{npq}}+\dfrac{pq^2\theta^2}{2npq}+\cdots\cdots)^n=$ $(1+\dfrac{\theta^2}{2n}+\cdots\cdots)^n\to$ $e^{\theta^2/2}$	M1 M1 1 1	For expansion of exponential terms For indication that these can be neglected as $n\to\infty$. Use of result given in question	4

(vi)	N(0,1) Because $e^{\theta^2/2}$ is the mgf of N(0,1) and the relationship between distributions and their mgfs is unique	1 E1 E1		3
(vii)	"Unstandardising", $\mathrm{N}(\mu, \sigma^2)$ ie $\mathrm{N}(np, npq)$	1	Parameters need to be given.	1

3(i)	$H_0: \mu_A = \mu_B$ $H_1: \mu_A \neq \mu_B$ Where μ_A, μ_B are the population means Test statistic $\dfrac{26.4 - 25.38}{\sqrt{\dfrac{2.45}{7} + \dfrac{1.40}{5}}} =$ $\dfrac{1.02}{\sqrt{0.63} = 0.7937} = 1.285$ Refer to N(0,1) Double-tailed 5% point is 1.96 Not significant No evidence that the population means differ	1 1 M1 M1 M1 A1 1 1 1 1	Do NOT allow $\overline{X} = \overline{Y}$ or similar Accept absence of "population" if correct notation μ is used. Hypotheses stated verbally must include the word "population". Numerator Denominator two separate terms correct No FT if wrong No FT if wrong	10
(ii)	CI (for $\mu_A - \mu_B$) is $1.02 \pm$ $1.645 \times$ $0.7937 =$ $1.02 \pm 1.3056 =$ (– 0.2856, 2.3256)	M1 B1 M1 A1 cao	Zero out of 4 if not N(0,1)	4
(iii)	H_0 is accepted if –1.96< test statistic < 1.96 i.e. if $-1.96 < \dfrac{\bar{x} - \bar{y}}{0.7937} < 1.96$ i.e. if $-1.556 < \bar{x} - \bar{y} < 1.556$ In fact, $\overline{X} - \overline{Y} \sim \mathrm{N}(2, 0.7937^2)$ So we want $\mathrm{P}(-1.556 < \mathrm{N}(2, 0.7937^2) < 1.556) =$ $\mathrm{P}\left(\dfrac{-1.556 - 2}{0.7937} < \mathrm{N}(0,1) < \dfrac{1.556 - 2}{0.7937}\right) =$ $\mathrm{P}(-4.48 < \mathrm{N}(0,1) < -0.5594) = 0.2879$	M1 M1 A1 M1 M1 M1 A1 cao	SC1 Same wrong test can get M1,M1,A0. SC2 Use of 1.645 gets 2 out of 3. BEWARE PRINTED ANSWER Standardising	7
(iv)	Wilcoxon would give protection if assumption of Normality is wrong. Wilcoxon could not really be applied if underlying variances are indeed different. Wilcoxon would be less powerful (worse Type II error behaviour) with such small samples if Normality is correct.	E1 E1 E1		3

<table>
<tr><td>4 (i)</td><td>There might be some consistent source of plot-to-plot variation that has inflated the residual and which the design has failed to cater for.</td><td>E2</td><td>E1 – Some reference to extra variation.
E1 – Some indication of a reason.</td><td>2</td></tr>
<tr><td>(ii)</td><td>Variation between the fertilisers should be compared with experimental error.

If the residual is inflated so that it measures more than experimental error, the comparison of between - fertilisers variation with it is less likely to reach significance.</td><td>E1

E2</td><td>

(E1, E1)</td><td>3</td></tr>
<tr><td>(iii)</td><td>Randomised blocks
<table><tr><td>C
B
A
D
E</td><td>.
.
.</td><td>.
.
.</td></tr></table>
SPECIAL CASE: Latin Square $\frac{2}{4}$ (1, E1)</td><td>1
E1
E1
E1</td><td>Blocks (strips) clearly correctly oriented w.r.t. fertiliser gradient.
All fertilisers appear in a block.
Different (random) arrangements in the blocks.</td><td>4</td></tr>
<tr><td>(iv)</td><td>Totals are: 95.0 123.2 86.8 130.2 67.4
(each from sample of size 4)
Grand total 502.6
"Correction factor" CF = $\frac{502.6^2}{20} = 12630.338$
Total SS = 13610.22 – CF = 979.882

Between fertilisers SS =
$\frac{95.0^2}{4}+...+\frac{67.4^2}{4}$ – CF =
13308.07 – CF = 677.732

Residual SS (by subtraction) =
979.882 – 677.732=302.15

<table><tr><th>Source of variation</th><th>SS</th><th>df</th><th>MS</th><th>MS Ratio</th></tr><tr><td>Between fertiliser</td><td>677.732</td><td>4</td><td>169.433</td><td>8.41</td></tr><tr><td>Residual</td><td>302.15</td><td>15</td><td>20.143</td><td></td></tr><tr><td>Total</td><td>979.882</td><td>19</td><td></td><td></td></tr></table>
Refer to $F_{4, 15}$
-upper 5% point is 3.06
Significant
- seems effects of fertilisers are not all the same</td><td>M1
M1
A1

M1
M1
1,A1
1

1
1
1
1</td><td>
For correct method for any two
If each calculated SS is correct

No FT if wrong
No FT if wrong</td><td>12</td></tr>
<tr><td>(vii)</td><td>Independent N (0, σ^2 [constant])</td><td>1
1
1</td><td></td><td>3</td></tr>
</table>

4769 Statistics 4

Q1				
(i)	$L = \frac{e^{-\theta}\theta^{x_1}}{x_1!}\cdots\frac{e^{-\theta}\theta^{x_n}}{x_n!}\left[= \frac{e^{-n\theta}\theta^{\sum x_i}}{x_1!x_2!\cdots x_n!}\right]$	M1 A1	product form fully correct	
	ln L = const − $n\theta + \sum x_i \ln\theta$	M1 A1		
	$\frac{d\ln L}{d\theta} = -n + \frac{\sum x_i}{\theta} = 0$	M1 A1		
	$\Rightarrow \hat{\theta} = \frac{\sum x_i}{n} (= \bar{x})$	A1	CAO	
	Check this is a maximum	M1		
	e.g. $\frac{d^2\ln L}{d\theta^2} = -\frac{\sum x_i}{\theta^2} < 0$	A1		9
(ii)	$\lambda = P(X=0) = e^{-\theta}$	B1		1
(iii)	We have $R \sim B(n, e^{-\theta})$, so $E(R) = ne^{-\theta}$ $Var(R) = ne^{-\theta}(1-e^{-\theta})$	M1 B1 B1		
	$\tilde{\lambda} = \frac{R}{n}$	M1		
	$\therefore E(\tilde{\lambda}) = e^{-\theta}$ i.e. unbiased	A1 A1		
	$Var(\tilde{\lambda}) = \frac{e^{-\theta}(1-e^{-\theta})}{n}$	A1	BEWARE PRINTED ANSWER	7

(iv)	Relative efficiency of $\tilde{\lambda}$ wrt ML est $= \dfrac{\text{Var(ML Est)}}{\text{Var}(\tilde{\lambda})}$	M1	any attempt to compare variances	
		M1	if correct	
	$= \dfrac{\theta e^{-2\theta}}{n} \cdot \dfrac{n}{e^{-\theta}(1-e^{-\theta})} = \dfrac{\theta}{e^{\theta}-1}$	A1	BEWARE PRINTED ANSWER	
	Eg:- Expression is $\dfrac{\theta}{\theta + \dfrac{\theta^2}{2!} + \ldots}$	M1		
	always < 1	E1		
	and this is $\approx$ 1 if θ is small	E1		
	$\approx$ 0 if θ is large	E1	Allow statement that $\dfrac{\theta}{e^{\theta}-1} \to 0 \text{ as } \theta \to \infty$	
				7

Q2				
(i)	$\text{P}(X=x)=q^{x-1}p$	B1	FT into pgf only	
	Pgf $\text{G}(t)=\text{E}(t^X)=\sum_{x=1}^{\infty}pt^xq^{x-1}$	M1		
	$=pt(1+qt+q^2t^2+\ldots)$	A1		
	$=\underline{\underline{pt(1-qt)^{-1}}}$	A1	BEWARE PRINTED ANSWER [consideration of |qt| < 1 not required]	
	$\mu=\text{G}'(1)\quad \sigma^2=\text{G}''(1)+\mu-\mu^2$	M1	for attempt to find G'(t) and/or G"(t)	
	$G'(t)=pt(-1)(1-qt)^{-2}(-q)+p(1-qt)^{-1}$ $=pqt(1-qt)^{-2}+p(1-qt)^{-1}$	A1		
	$\therefore \text{G}'(1)=pq(1-q)^{-2}+p(1-q)^{-1}=\frac{q}{p}+1=\underline{\underline{\frac{1}{p}}}$	A1	BEWARE PRINTED ANSWER	
	$\text{G}''(t)=pqt(-2)(1-qt)^{-3}(-q)+pq(1-qt)^{-2}+$ $p(-1)(1-qt)^{-2}(-q)$	A1		
	$\therefore \text{G}''(1)=2pq^2(1-q)^{-3}+pq(1-q)^{-2}+pq(1-q)^{-2}$ $=\frac{2q^2}{p^2}+\frac{2q}{p}$	A1		
	$\therefore \sigma^2=\frac{2q^2}{p^2}+\frac{2q}{p}+\frac{1}{p}-\frac{1}{p^2}=\frac{2q^2+2pq+p-1}{p^2}$	M1	For inserting their values	
	$=\frac{q}{p^2}(2q+2p-1)=\underline{\underline{\frac{q}{p^2}}}$	A1	BEWARE PRINTED ANSWER	
				11

(ii)	X_1=number of trials to first success X_2= " " " " " next " . . . . X_n= " " " " " nth " } $\therefore Y=X_1+X_2+\ldots..X_n$ = total no of trials to the nth success $\therefore$ pgf of $Y = (\text{pgf of } X)^n = \underline{\underline{p^n t^n (1-qt)^{-n}}}$ $\mu_Y = n\mu_X = \underline{\underline{\dfrac{n}{p}}}$ $\sigma_Y^2 = n\sigma_X^2 = \underline{\underline{\dfrac{nq}{p^2}}}$	E1 E1 1 1 1		5
(iii)	N(candidate's μ_Y , candidate's σ_Y^2)	1		1
(iv)	Y = no of tickets to be sold ~ random variable as in (ii) with n = 140 and p = 0.8 ~ Approx N($\dfrac{140}{0.8} = 175, \dfrac{140 \times 0.2}{(0.8)^2} = 43.75$) $P(Y \geq 160) \approx P(N(175, 43.75) > 159\frac{1}{2})$ = P(N(0,1)>-2.343) = 0.9905 For any sensible discussion in context (eg groups of passengers ⇒ not indep.)	E1 1 M1 A1 A1 E1 E1	Do not award if cty corr absent or wrong, but FT if 160 used → -2.268, 0.9884 CAO	7
Q3	X = amount of salt ~ N($\mu[750], \sigma^2[20^2]$) Sample of n=9			
(i)	Type I error: rejecting null hypothesis … … when it is true. Type II error: accepting null hypothesis … … when it is false. OC: P (accepting null hypothesis … … as a function of the parameter under investigation)	B1 B1 B1 B1 B1 B1	Allow B1 for P(rej H_0 when true) Allow B1 for P(acc H_0 when false) [P(type II error | the true value of the parameter) scores B1+B1]	6
(ii)	Reject if $\bar{x} < 735 \text{ or } \bar{x} > 765$ $\alpha = P(\bar{X} < 735 \text{ or } \bar{X} > 765 \mid \bar{X} \sim N(750, \frac{20^2}{9}))$ $= P(Z < \dfrac{(735-750)3}{20} = -2.25$ $\text{or } Z > \dfrac{(765\text{-}750)3}{20} = 2.25)$ = 2(1-0.9878) = 2 × 0.0122 = 0.0244 This is the probability of rejecting good output and unnecessarily re-calibrating the machine – seems small [but not very small?]	M1 A1 A1 A1 E1 E1	Might be implicit CAO Accept any sensible comments	6

(iii)	Accept if $735 < \bar{x} < 765$, and now $\mu = 725$. $\beta = P(735 < \overline{X} < 765 \mid \overline{X} \sim N(725, 20^2/9))$ = P(1.5 < Z< 6) = 1 – 0.9332 = 0.0668 This is the probability of accepting output and carrying on when in fact μ has slipped to 725 – small[-ish?]	M1 A1 A1 A1 E1 E1	might be implicit CAO If upper limit 765 not considered, maximum 2 of these 4 marks. If Φ(6) not considered, maximum 3 out of 4. accept sensible comments	6
(iv)	OC = $P(735 < \overline{X} < 765 \mid \overline{X} \sim N(\mu, 20^2/9))$ $= \Phi\left(\frac{(765-\mu)3}{20}\right) - \Phi\left(\frac{(735-\mu)3}{20}\right)$ "Φ − Φ" μ=720:Φ(6.75) – Φ(2.25)=1– 0.9878 =0.0122 730: 5.25 0.75 =1– 0.7734 =0.2266 740: 3.75 −0.75 =1– (1–0.7734)= 0.7734 750: similarly or by write-down from part (ii) [FT] : 0.9756 760, 770, 780 by symmetry [FT]: 0.7734, 0.2266, 0.0122	M1 M1 A1 1 1 1	 both correct if any two correct	6
Q4				
(i)	$x_{ij} = \mu + \alpha_i + e_{ij}$ μ = population … .. grand mean for whole experiment α_i = population … .. mean by which *i* th treatment differs from μ e_{ij} are experimental errors… ~ ind N $(0, \sigma^2)$	1 1 1 1 1 1 3	Allow "uncorrelated" 1 for ind N; 1 for 0; 1 for σ^2.	9
(ii)	Totals are 240, 246, 254, 264, 196 each from sample of size 5 Grand total 936 "Correction factor" CF = $\frac{936^2}{20}$ = 43804.8 Total SS = 44544 - CF = 739.2			

	Between contractors SS = $\frac{240^2}{5}+\ldots+\frac{196^2}{5}-\text{CF}$ = 44209.6 – CF = 404.8 Residual SS (by subtraction) = 739.2 – 404.8 = 334.4	M1 M1 A1	For correct methods for any two, if each calculated SS is correct.	
	ANOVA table: Source of Variation | SS | df | MS | MS ratio Between Contractors | 404.8 | 3 | $134.9\dot{3}$ | 6.456 Residual | 334.4 | 16 | 20.9 | Total | 739.2 | 19 | |	M1 (MS) M1 (MS ratio) 1 (MS) A1 (6.456) 1 (df 19)	CAO	
	Refer to $F_{3,16}$ Upper 5% point is 3.24 Significant Seems performances of contractors are not all the same	1 1 1 1	NO FT IF WRONG NO FT IF WRONG	12
(iii)	Randomised blocks Description	B1 E1 E1	Take the subject areas as "blocks", ensure each contractor is used at least once in each block	3

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Q1	Follow-through all intermediate results in this question, unless obvious nonsense.			
(i)	$P(X \geq 2) = 1 - \theta - \theta(1-\theta) = (1-\theta)^2$ [o.e.] $L = [\theta]^{n_0}[\theta(1-\theta)]^{n_1}[(1-\theta)^2]^{n-n_0-n_1}$ $= \theta^{n_0+n_1}(1-\theta)^{2n-2n_0-n_1}$	M1 A1 M1 A1 A1	 Product form Fully correct BEWARE PRINTED ANSWER	5
(ii)	$\ln L = (n_0+n_1)\ln\theta + (2n-2n_0-n_1)\ln(1-\theta)$ $\frac{d\ln L}{d\theta}$ $= \frac{n_0+n_1}{\theta} - \frac{2n-2n_0-n_1}{1-\theta}$ $= 0$ $\Rightarrow (1-\hat{\theta})(n_0+n_1) = \hat{\theta}(2n-2n_0-n_1)$ $\Rightarrow \hat{\theta} = \frac{n_0+n_1}{2n-n_0}$	M1 A1 M1 A1 M1 A1		6
(iii)	$E(X) = \sum_{x=0}^{\infty} x\theta(1-\theta)^x$ $= \theta\{0 + (1-\theta) + 2(1-\theta)^2 + 3(1-\theta)^3 + \ldots\}$ $= \frac{1-\theta}{\theta}$ So could sensibly use (method of moments) $\tilde{\theta}$ given by $\frac{1-\tilde{\theta}}{\tilde{\theta}} = \bar{X}$ $\Rightarrow \tilde{\theta} = \frac{1}{1+\bar{X}}$ To use this, we need to know the exact numbers of faults for components with "two or more".	M1 A2 M1 A1 E1	 Divisible, for algebra; e.g. by "GP of GPs" BEWARE PRINTED ANSWER BEWARE PRINTED ANSWER	 6
(iv)	$\bar{x} = \frac{14}{100} = 0 \cdot 14$ $\tilde{\theta} = \frac{1}{1+0.14} = 0{\cdot}8772$ Also, from expression given in question, $\text{Var}(\tilde{\theta}) = \frac{0 \cdot 8772^2(1-0 \cdot 8772)}{100}$ $= 0{\cdot}000945$ CI is given by $0{\cdot}8772 \pm 1{\cdot}96 \times \sqrt{0 \cdot 000945}$ = (0·817, 0·937)	B1 B1 B1 M1 B1 M1 A1	 For 0.8772 For 1.96 For $\sqrt{0 \cdot 000945}$	 7

Q2				
(i)	Mgf of Z = $E(e^{tZ}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tz - \frac{z^2}{2}} dz$	M1		
	Complete the square $tz - \frac{z^2}{2} = -\frac{1}{2}(z-t)^2 + \frac{1}{2}t^2$	M1 A1 A1		
	$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}} dt = e^{\frac{t^2}{2}}$ Pdf of N(t,1) $\therefore \int = 1$	M1 M1 M1 A1	For taking out factor $e^{\frac{t^2}{2}}$ For use of pdf of N(t,1) For $\int$ pdf = 1 For final answer $e^{\frac{t^2}{2}}$	8
(ii)	*Y* has mgf $M_Y(t)$ Mgf of *aY* + *b* is $E[e^{t(aY+b)}]$ $= e^{bt}E[e^{(at)Y}] = e^{bt}M_Y(at)$	M1 1 1 1	For factor e^{bt} For factor $E[e^{(at)Y}]$ For final answer	4
(iii)	$Z = \frac{X-\mu}{\sigma}$, so $X = \sigma Z + \mu$ $\therefore M_X(t) = e^{\mu t}.e^{\frac{(\sigma t)^2}{2}} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$	M1 1 1 1	For factor $e^{\mu t}$ For factor $e^{\frac{(\sigma t)^2}{2}}$ For final answer	4
(iv)	$W = e^X$ $E(W^k) = E[(e^X)^k] = E(e^{kX}) = M_X(k)$	M1 A1 A1	For $E[(e^X)^k]$ For $E(e^{kX})$ For $M_X(k)$	
	$\therefore E(W) = M_X(1) = e^{\mu + \frac{\sigma^2}{2}}$	M1 A1		
	$E(W^2) = M_X(2) = e^{2\mu + 2\sigma^2}$ $\therefore \text{Var}(W) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} [= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)]$	M1 A1 A1		8

Q3			
(i)	$\bar{x} = 126\cdot2$ s = 8·7002 $s^2 = 75\cdot69\dot{3}$ $\bar{y} = 133\cdot9$ s = 10·4760 $s^2 = 109\cdot74\dot{6}$	A1	A1 if all correct. [No mark for use of s_n, which are 8·2537 and 9·6989 respectively.]
	$H_0 : \mu_A = \mu_B$ $H_0 : \mu_A \neq \mu_B$ Where μ_A, μ_B are the population means.	1 1	<u>Do not</u> accept $\overline{X} = \overline{Y}$ or similar.
	Pooled s^2 $= \frac{9\times75\cdot69\dot{3} + 6\times109\cdot47\dot{6}}{15} = \frac{681\cdot24 + 658\cdot48}{15}$ $= 89\cdot314\dot{6}$ [√ = 9·4506]	B1	
	Test statistic is $\frac{126\cdot2 - 133\cdot9}{\sqrt{89\cdot314\dot{6}}\sqrt{\frac{1}{10}+\frac{1}{7}}} = -\frac{7\cdot7}{4\cdot6573} = -1\cdot653$	M1 A1	
	Refer to t_{15}	1	No FT if wrong
	Double-tailed 10% point is 1·753 Not significant No evidence that population mean concentrations differ.	1 1 1	No FT if wrong 10
(ii)	There may be consistent differences between days (days of week, types of rubbish, ambient conditions,…) which should be allowed for.	E1 E1	
	Assumption: Normality of population of <u>differences.</u>	1	
	Differences are 7·4 -1·2 11·1 5·5 6·2 3·7 −0·3 1·8 3·6 [$\bar{d}$ = 4·2, s = 3·862 (s^2 =14·915)] Use of $s_n (= 3\cdot641)$ is <u>not</u> acceptable, even in a denominator of $s_n / \sqrt{n-1}$]	M1	A1 Can be awarded here if NOT awarded in part (i)
	Test statistic is $\frac{4\cdot2 - 0}{3\cdot862/\sqrt{9}}$ = 3·26	M1 A1	
	Refer to t_8 Double-tailed 5% point is 2·306 Significant Seems population means differ	1 1 1 1	No FT if wrong No FT if wrong 10

(iii)	Wilcoxon rank sum test Wilcoxon signed rank test H_0: median$_A$ = median$_B$ H_1: median$_A$ ≠ median$_B$	B1 B1 1 1	[Or more formal statements]	4
Q4				
(i)	Description must be in <u>context</u>. If no context given, mark according to scheme and then give half-marks, rounded down. Clear description of "rows". And "columns" As extraneous factors to be taken account of in the design, with "treatments" to be compared. Need same numbers of each Clear contrast with situations for completely randomised design and randomised trends.	E1 E1 E1 E1 E1 E1 E1 E1 E1		9
(ii)	e_{ij} ~ ind N (0, σ^2) α_i is population mean effect by which *i*th treatment differs from overall mean	1 1 1 1 1	Allow uncorrelated For 0 For σ^2	5
(iii)	(ANOVA table below)	1 1 1 1 1 1 1		
	Refer to $F_{4,15}$	1	No FT if wrong	
	Upper 1% point is 4·89	1	No FT if wrong	
	Significant, seems treatments are not all the same	1		10

Source of Variation	SS	df	MS	MS ratio
Between Treatments	92·30	4	23·075	5·034
Residual	68·76	15	4·584	
Total	161·06	19		

Question 1

	$f(x) = \dfrac{xe^{-x/\lambda}}{\lambda^2} \quad (x > 0)$	
(i)	$E(X) = \dfrac{1}{\lambda^2}\int_0^\infty x^2 e^{-x/\lambda}dx$ $= \dfrac{1}{\lambda^2}\left\{\left[-\lambda x^2 e^{-x/\lambda}\right]_0^\infty + \int_0^\infty \lambda.2xe^{-x/\lambda}dx\right\}$ $= \dfrac{1}{\lambda^2}\{[0-0]\} + 2\lambda.1 = 2\lambda$. $E(\bar{X}) = E(X) \quad \therefore E\left(\hat{\lambda}[=\tfrac{1}{2}\bar{X}]\right) = \lambda \quad \therefore \hat{\lambda}$ is unbiased.	M1 for integral for E(X) M1 for attempt to integrate by parts For second term: M1 for use of integral of pdf or for integr'g by parts again A1 M1 A1 E1 **[7]**
(ii)	$\text{Var}(\hat{\lambda}) = \dfrac{1}{4}\text{Var}(\bar{X}) = \dfrac{1}{4}\dfrac{\text{Var}(X)}{n}$ $E(X^2) = \dfrac{1}{\lambda^2}\int_0^\infty x^3 e^{-x/\lambda}dx$ $= \dfrac{1}{\lambda^2}\left\{\left[-\lambda x^3 e^{-x/\lambda}\right]_0^\infty + \int_0^\infty 3\lambda x^2 e^{-x/\lambda}dx\right\}$ $= \dfrac{1}{\lambda^2}\{[0-0]\} + 3\lambda E(X) = 6\lambda^2$. $\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = 6\lambda^2 - 4\lambda^2 = 2\lambda^2$. $\therefore \text{Var}(\hat{\lambda}) = \dfrac{\lambda^2}{2n}$.	M1 M1 for use of E(X^2) By parts M1 M1 for use of E(X) A1 for $6\lambda^2$ A1 A1 **[7]**
(iii)	Variance of $\hat{\lambda}$ becomes very small as n increases. It is unbiased and so becomes increasingly concentrated at the correct value λ.	E1 E1 **[2]**
(iv)	$E(\tilde{\lambda}) = \left(\tfrac{1}{8} + \tfrac{1}{4} + \tfrac{1}{8}\right)2\lambda = \lambda$. $\quad \therefore \tilde{\lambda}$ is unbiased. $\text{Var}(\tilde{\lambda}) = \left(\tfrac{1}{64} + \tfrac{1}{16} + \tfrac{1}{64}\right)2\lambda^2 = \tfrac{3}{16}\lambda^2$. $\therefore$ relative efficiency of $\tilde{\lambda}$ to $\hat{\lambda}$ is $\dfrac{\lambda^2/6}{3\lambda^2/16} = \dfrac{8}{9}$. **Special case.** If done as Var($\tilde{\lambda}$) / Var($\hat{\lambda}$), award 1 out of 2 for the second M1 and the A1 in the scheme. So $\hat{\lambda}$ is preferred.	$E(\tilde{\lambda})$: B1; "unbiased": E1 M1 A1 M1 any comparison of variances M1 correct comparison A1 for 8/9 [Note. This M1M1A1 is allowable in full as FT if everything is plausible.] E1 (FT from above) **[8]**

Question 2

(i) $\mathrm{G}(t)=\mathrm{E}(t^X)=\sum_{x=0}^{\infty}\frac{e^{-\lambda}(\lambda t)^x}{x!}$ **[M1]** $=e^{-\lambda}\left(1+\lambda t+\frac{\lambda^2t^2}{2!}+...\right)$ **[A1]** $=\mathrm{e}^{-\lambda}\mathrm{e}^{\lambda t}=\mathrm{e}^{\lambda(t-1)}$ **[A1]** **[Allow omission of previous A1 step and write-down of this for A2 provided opening M1 has been earned (NB answer is given)]**	**[3]**
(ii) Mean = G'(1) $\mathrm{G}'(t)=\lambda\mathrm{e}^{\lambda(t-1)}$ **[M1]** G'(1) = λ **[A1]** Variance = G"(1) + mean – mean^2 $\mathrm{G}''(t)=\lambda^2\mathrm{e}^{\lambda(t-1)}$ **[M1]** G"(1) = λ^2 **[A1]** $\therefore$ variance = $\lambda^2+\lambda-\lambda^2=\lambda$ **[A1]**	**[5]**
(iii) $Z=\frac{X-\mu}{\sigma}$: mean 0 **[B1]** variance 1 **[B1]**	**[2]**
(iv) Mgf of X is $\mathrm{M}(\theta)=\mathrm{G}(\mathrm{e}^{\theta})=\mathrm{e}^{\lambda(\mathrm{e}^{\theta}-1)}$ **[B1]** Linear transformation result is $\mathrm{M}_{aX+b}(\theta)=\mathrm{e}^{b\theta}\mathrm{M}_X(a\theta)$ **[B2 if fully correct, any equivalent form. Allow B1 if either factor correct.]** Use with $a=\frac{1}{\sigma}=\frac{1}{\sqrt{\lambda}}$ and $b=-\frac{\mu}{\sigma}=-\sqrt{\lambda}$ **[M1]** $\mathrm{M}_Z(\theta)=\mathrm{e}^{-\sqrt{\lambda}\theta}\mathrm{e}^{\lambda\left(\mathrm{e}^{\theta/\sqrt{\lambda}}-1\right)}=\mathrm{e}^{\lambda\left(\mathrm{e}^{\theta/\sqrt{\lambda}}-\frac{\theta}{\sqrt{\lambda}}-1\right)}$ **[A1]** **[A1]** **[A1]** **[NB answer is given]**	**[7]**
(v) Consider $\lambda\left(\mathrm{e}^{\theta/\sqrt{\lambda}}-\frac{\theta}{\sqrt{\lambda}}-1\right)=\lambda\left(1+\frac{\theta}{\sqrt{\lambda}}+\frac{\theta^2}{2!\lambda}+\frac{\theta^3}{3!\lambda^{3/2}}+...-\frac{\theta}{\sqrt{\lambda}}-1\right)$ **[M1]** $=\frac{\theta^2}{2}+$ terms in $\lambda^{-1/2},\lambda^{-1},\lambda^{-3/2},...$ **[A1]** $\rightarrow\frac{\theta^2}{2}$ as $\lambda\to\infty$ **[M1]** **[some explanation required]** $\therefore\mathrm{M}_Z(\theta)\to\mathrm{e}^{\theta^2/2}$ as $\lambda\to\infty$ **[A1] [answer given]**	**[4]**
(vi) $\mathrm{e}^{\theta^2/2}$ is the mgf of N(0, 1) **[E1]**, and the relationship between distributions and their mgfs is unique **[E1]**. "Unstandardising", X tends to N(μ, σ^2) i.e. N(λ, λ) **[B1, parameters must be given]**.	**[3]**

Question 3

(i) H_0 is accepted if –1.96 < value of test statistic < 1.96 i.e. if $-1.96 < \dfrac{(\bar{x}_1-\bar{x}_2)-(0)}{\sqrt{\dfrac{1.2^2}{8}+\dfrac{1.4^2}{10}}} < 1.96$ i.e. if $-1.96\times0.6132 < \bar{x}_1-\bar{x}_2 < 1.96\times0.6132$ i.e. if $-1.20(18) < \bar{x}_1-\bar{x}_2 < 1.20(18)$ **Note.** Use of $\mu_1 - \mu_2$ instead of $\bar{x}_1-\bar{x}_2$ can score M1 B1 M0 M1 A0 A0.	M1 double inequality B1 1.96 M1 num[r] of test statistic M1 den[r] of test statistic A1 A1 **Special case.** Allow 1 out of 2 of the A1 marks if 1.645 used provided all 3 M marks have been earned. **[6]**
(ii) $\bar{x}_1-\bar{x}_2=1.4$ which is outside the acceptance region so H_0 is rejected. CI for $\mu_1 - \mu_2$: 1.4 ± (2.576 × 0.6132), i.e. 1.4 ± 1.5796, i.e. (–0.18 [–0.1796] , 2.97[96])	B1 FT if wrong M1 [FT can's acceptance region if reasonable] E1 M1 for 1.4 B1 for 2.576 M1 for 0.6132 A1 cao for interval **[7]**
(iii) Wilcoxon rank sum test (or Mann-Whitney form of test) Ranks are: First 14 13 10 8 6 11 Second 2 12 3 1 4 7 5 9 *W* = 14 + 13 + 10 + 8 + 6 + 11 = 62 [or 8 + 8 + 7 + 7 + 6 + 5 = 41 if M-W used] Refer to $W_{6,8}$ [or $MW_{6,8}$] tables. Lower 2½% critical point is 29 [or 8 if M-W used]. Consideration of upper 2½% point is also needed. Eg: by using symmetry about mean of $(\frac{1}{2}\times6\times8)+(\frac{1}{2}\times6\times7)$ = 45, critical point is 61. [For M-W: mean is $\frac{1}{2}\times6\times8$ = 24, hence critical point is 40.] Result is significant. Seems (population) medians may not be assumed equal.	M1 M1 Combined ranking A1 Correct [allow up to 2 errors; FT provided M1 earned] B1 M1 No FT if wrong A1 **Special case 1.** If M1 for $W_{6,8}$ has not been awarded (likely to be because rank sum 43 has been used, which should be referred to $W_{8,6}$), the next two M1 marks can be earned but *nothing beyond them*. M1 M1 for any correct method A1 if 61 correct E1, E1 **Special case 2** (does not apply if Special Case 1 has been invoked). These 2 marks may be earned even if only 1 or 2 of the preceding 3 have been earned. **[11]**

Question 4

(i)	Randomised blocks Eg:- WEST | D A C B | C B A D | D C A B | EAST Plots in strips (blocks) correctly aligned w.r.t. fertility trend. Each letter occurs at least once in each block in a random arrangement.	B1 M1 E1 M1 E1 **[5]**
(ii)	μ = population **[B1]** grand mean for whole experiment **[B1]** α_i = population **[B1]** mean amount by which the *i*th treatment differs from μ **[B1]** e_{ij} ~ ind N [**B1**, accept "uncorrelated"] (0 **[B1]** , σ^2 **[B1]**)	4 marks, as shown 3 marks, as shown **[7]**
(ii)	Totals are 62.7 65.6 69.0 67.8 all from samples of size 5 Grand total 265.1 "Correction factor" CF = $265.1^2/20$ = 3513.9005 Total SS = 3524.31 – CF = 10.4095 Between varieties SS = $\frac{62.7^2}{5}+\frac{65.6^2}{5}+\frac{69.0^2}{5}+\frac{67.8^2}{5}$ – CF = 3518.498 – CF = 4.5975 Residual SS (by subtraction) = 10.4095 – 4.5975 = 5.8120	M1 for attempt to form three sums of squares. M1 for correct method for any two. A1 if each calculated SS is correct.
	(see table below)	5 marks within the table, as shown
	Refer MS ratio to $F_{3,16}$.	M1 No FT if wrong
	Upper 5% point is 3.24. Significant. Seems the mean yields of the varieties are not all the same.	A1 No FT if wrong E1 E1 **[12]**

Source of variation	SS	df	MS **[M1]**	MS ratio **[M1]**
Between varieties	4.5975	3 **[B1]**	1.5325	4.22 **[A1 cao]**
Residual	5.8120	16 **[B1]**	0.36325	
Total	10.4095	19		

4769 June 2011 Qu 1

$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x^2/2\theta}$ [N(0, θ)]	
(i) $L = \frac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x_1^2/2\theta} \cdot \frac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x_2^2/2\theta} \cdot ... \cdot \frac{1}{\sqrt{2\pi\theta}} \mathrm{e}^{-x_n^2/2\theta}$	M1 product form A1 fully correct
$\left[= (2\pi\theta)^{-n/2} \mathrm{e}^{-\Sigma x_i^2/2\theta} \right]$	Note. This A1 mark and the next five A1 marks depend on *all* preceding M marks having been earned.
$\ln L = -\frac{n}{2}\ln(2\pi\theta) - \frac{1}{2\theta}\sum x_i^2$	M1 for ln L A1 fully correct
$\frac{\mathrm{d}\ln L}{\mathrm{d}\theta} = -\frac{n}{2}\cdot\frac{1}{\theta} + \frac{1}{2\theta^2}\sum x_i^2$	M1 for differentiating A1, A1 for each term
$\frac{\mathrm{d}\ln L}{\mathrm{d}\theta} = 0$ gives $\frac{n}{2\hat{\theta}} = \frac{1}{2\hat{\theta}^2}\sum x_i^2$	M1 A1
i.e. $\hat{\theta} = \frac{1}{n}\sum x_i^2$	A1
Check this is a maximum. Eg:	M1
$\frac{\mathrm{d}^2\ln L}{\mathrm{d}\theta^2} = \frac{n}{2}\cdot\frac{1}{\theta^2} - \frac{1}{\theta^3}\sum x_i^2$	A1
which, for $\theta = \hat{\theta}$, is $\frac{n}{2\hat{\theta}^2} - \frac{n}{\hat{\theta}^2} = -\frac{n}{2\hat{\theta}^2} < 0$.	A1 for expression involving $\hat{\theta}$ A1 for showing < 0 **[14]**
(ii) First consider E(X^2) = Var(X) + {E(X)}^{2} = θ + 0	M1 A1
$\therefore \mathrm{E}(\hat{\theta}) = \frac{1}{n}(\theta + \theta + ... + \theta) = \theta$	A1
i.e. $\hat{\theta}$ is unbiased.	A1 **[4]**
(iii) Here $\hat{\theta} = 10$ and Est Var$(\hat{\theta})$ = 2 × 10^2/100 = 2	B1, B1
Approximate confidence interval is given by $10 \pm 1.96\sqrt{2} = 10 \pm 2.77$, i.e. it is (7.23, 12.77).	M1 centred at 10 B1 1.96 M1 Use of √2 A1 c.a.o. Final interval **[6]**

4769 June 2011 Qu 2

(i) $n = 2$ $\quad \mathrm{f}(x) = \frac{1}{2}\mathrm{e}^{-x/2}$	
$\mathrm{M}(\theta) = \mathrm{E}(\mathrm{e}^{\theta X}) = \int_0^\infty \frac{1}{2}\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}\mathrm{d}x$	A1 Any equivalent form
$= \frac{1}{2}\left[\frac{e^{-x\left(\frac{1}{2}-\theta\right)}}{-\left(\frac{1}{2}-\theta\right)}\right]_0^\infty$ **[A1]** $= \frac{\frac{1}{2}}{\frac{1}{2}-\theta}$ **[A1]** $= (1-2\theta)^{-1}$ **[A1]**	A1, A1, A1 for each expression, as shown, **beware printed answer**
$n = 4$ $\quad \mathrm{f}(x) = \frac{1}{4}x\mathrm{e}^{-x/2}$	
$\mathrm{M}(\theta) = \int_0^\infty \frac{1}{4}x\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}\mathrm{d}x$	M1 for attempt to integrate this by parts
$= \frac{1}{4}\left\{\left[\frac{x\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}}{-\left(\frac{1}{2}-\theta\right)}\right]_0^\infty \textbf{[A1]} - \int_0^\infty \frac{\mathrm{e}^{-x\left(\frac{1}{2}-\theta\right)}}{-\left(\frac{1}{2}-\theta\right)}\mathrm{d}x \textbf{ [A1]}\right\}$	A1, A1 for each component, as shown
$= \frac{1}{4}\left\{[0-0] \textbf{ [A1]} + \frac{1}{\frac{1}{2}-\theta}.2(1-2\theta)^{-1} \textbf{ [A1]}\right\}$	A1, A1 for each component, as shown
$= \frac{1}{2}\frac{1}{\frac{1}{2}(1-2\theta)}(1-2\theta)^{-1} = (1-2\theta)^{-2}$	A1 for final answer, **beware printed answer**
	[10]
(ii) Mean = M'(0) $\quad \mathrm{M}'(\theta) = -2\left(-\frac{n}{2}\right)(1-2\theta)^{-\frac{n}{2}-1} = n(1-2\theta)^{-\frac{n}{2}-1}$	M1 A1
$\therefore$ mean = n	A1
Variance = M''(0) – {M'(0)}^{2}	
$\mathrm{M}''(\theta) = n\left(-\frac{n}{2}-1\right)(-2)(1-2\theta)^{-\frac{n}{2}-2} = n(n+2)(1-2\theta)^{-\frac{n}{2}-2}$	M1 A1
$\therefore$ M''(0) = $n(n + 2)$	A1
$\therefore$ variance = $n(n + 2) - n^2 = 2n$	A1
[**Note.** This part of the question may also be done by expanding the mgf.]	**[7]**

Solution continued on next page

4769 June 2011 Qu 2 **continued**

(iii)	By convolution theorem, $$M_W(\theta) = \left\{(1-2\theta)^{-\frac{1}{2}}\right\}^k = (1-2\theta)^{-k/2}.$$ This is the mgf of χ^2_k, so (by uniqueness of mgfs) $W \sim \chi^2_k$.	M1 B1 M1 B1 **[4]**
(iv)	$W \sim \chi^2_{100}$ has mean 100, variance 200. Can regard W as the sum of a large "random sample" of χ^2_1 variates. $$\therefore P\left(\chi^2_{100} < 118.5\right) \approx P\left(N(0,1) < \frac{118.5-100}{\sqrt{200}} = 1.308\right)$$ = 0.9045.	M1 for use of N(0,1) A1 c.a.o. for 1.308 A1 c.a.o. **[3]**

4769 June 2011 Qu 3

(i)	Type I error: rejecting null hypothesis **[B1]** when it is true **[B1]** Type II error: accepting null hypothesis **[B1]** when it is false **[B1]** OC: P(accepting null hypothesis **[B1]** as a function of the parameter under investigation **[B1]**) Power: P(rejecting null hypothesis **[B1]** as a function of the parameter under investigation **[B1]**)	8 separate B1 marks for components of answer, as shown Allow B1 out of 2 for P(...) Allow B1 out of 2 for P(...) P(Type II error | the true value of the parameter) scores B1+B1 P(Type I error | the true value of the parameter) scores B1+B1. "1 – OC" as definition scores zero. **[8]**
(ii)	$X \sim N(\mu, 25)$ H$_0$: $\mu = 94$ H$_1$: $\mu > 94$ We require $0.02 = P(\text{reject } H_0 \mid \mu = 94) = P(\bar{X} > c \mid \mu = 94)$ $= P(N(94, 25/n) > c) = P\left(N(0,1) > \frac{c-94}{5/\sqrt{n}}\right)$ $\therefore \frac{c-94}{5/\sqrt{n}} = 2.054$ We also require $0.95 = P(\text{reject } H_0 \mid \mu = 97)$ $= P(N(97, 25/n) > c) = P\left(N(0,1) > \frac{c-97}{5/\sqrt{n}}\right)$ $\therefore \frac{c-97}{5/\sqrt{n}} = -1.645$ $\therefore$ we have $c = 94 + \frac{10.27}{\sqrt{n}}$ and $c = 97 - \frac{8.225}{\sqrt{n}}$ Attempt to solve; c = 95.666 [allow 95.7 or awrt] $\sqrt{n} = 6.165$, $n = 38.01$ Take n as "next integer up" from candidate's value	M1 M1 for first expression M1 for standardising B1 for 2.054 M1 for first expression M1 for standardising B1 for –1.645 M1 two equations A1 both correct (FT any previous errors) M1 A1 c.a.o. A1 c.a.o. A1 **[13]**
(iii)	Power function: step function from 0 with step marked at 94 to height marked as 1	G1 G1 G1 Zero out of 3 if step is wrong way round. **[3]**

4769 June 2011 Qu 4

(a)	Each E2 in this part is available as E2, E1, E0.	
(i)	Description of situation where randomised blocks would be suitable, ie one extraneous factor (eg stream down one side of a field).	E2
	Explanation of why RB is suitable (the design allows the extraneous factor to be "taken out "separately).	E2
	Explanation of why LS is not appropriate (eg: there is only one extraneous factor; LS would be unnecessarily complicated; not enough degrees of freedom would remain for a sensible estimate of experimental error).	E2
(ii)	Description of situation where Latin square would be suitable, ie two extraneous factors (and all with same number of levels) (eg streams down two sides of a field).	E2
	Explanation of why LS is suitable (the design allows the extraneous factors to be "taken out "separately).	E2
	Explanation of why RB is not appropriate (RB cannot cope with two extraneous factors).	E2 **[12]**
(b)	Totals are 56.5 57.4 60.6 82.3 from samples of sizes 4 3 5 4 Grand total 256.8 "Correction factor" CF = $256.8^2/16$ = 4121.64 Total SS = 4471.92 – CF = 350.28 Between treatments SS = $\frac{56.5^2}{4}+\frac{57.4^2}{3}+\frac{60.6^2}{5}+\frac{82.3^2}{4}-\text{CF}$ = 4324.1103 – CF = 202.47 Residual SS (by subtraction) = 350.28 – 202.47 = 147.81	M1 for attempt to form three sums of squares. M1 for correct method for any two. A1 if each calculated SS is correct.
	(table below)	5 marks within the table, as shown
	Refer MS ratio to $F_{3,12}$. Upper 5% point is 3.49. Significant. Seems the effects of the treatments are not all the same.	M1 No FT if wrong A1 No FT if wrong E1 E1 **[12]**

Source of variation	SS	df	MS **[M1]**	MS ratio **[M1]**
Between treatments	202.47	3 **[B1]**	67.49	5.47(92) **[A1 cao]**
Residual	147.81	12 **[B1]**	12.3175	
Total	350.28	15		

Question			Answer	Marks	Guidance
1	(i)		$\mathrm{P}\left(X \le x\right) = \mathrm{F}_B(x).\frac{1}{2} + \mathrm{F}_G(x).\frac{1}{2}$ ie cdf of X is $\mathrm{F}\left(x\right) = \frac{1}{2}\left\{\mathrm{F}_B(x) + \mathrm{F}_G(x)\right\}$ ie (by differentiating) pdf of X is $\mathrm{f}\left(x\right) = \frac{1}{2}\left\{\mathrm{f}_B(x) + \mathrm{f}_G(x)\right\}$	M1 A1 A1 **[3]**	use of cdfs Answer given
1	(ii)		$\mathrm{E}\left(X\right)\left(= \frac{1}{2}\left\{\int x\mathrm{f}_B(x)\mathrm{d}x + \int x\mathrm{f}_G(x)\mathrm{d}x\right\}\right) = \frac{1}{2}\mu_B + \frac{1}{2}\mu_G$	M1 **[1]**	[answer given; needs *some* indication of method]
1	(iii)		$\mathrm{E}\left(X^2\right) = \int x^2\mathrm{f}(x)\mathrm{d}x$ $= \frac{1}{2}\left\{\int x^2\mathrm{f}_B(x)\mathrm{d}x + \int x^2\mathrm{f}_G(x)\mathrm{d}x\right\}$ Use of "$\mathrm{E}\left(X^2\right) = \sigma^2 + \mu^2$" $= \frac{1}{2}\left\{\sigma^2 + {\mu_B}^2 \quad + \quad \sigma^2 + {\mu_G}^2\right\}$ $\therefore \mathrm{Var}(X) = \mathrm{E}(X^2) - \left\{\mathrm{E}(X)\right\}^2$ $= \sigma^2 + \frac{1}{2}{\mu_B}^2 + \frac{1}{2}{\mu_G}^2 - \frac{1}{4}{\mu_B}^2 - \frac{1}{4}{\mu_G}^2 - \frac{1}{2}\mu_B\mu_G$ $= \sigma^2 + \frac{1}{4}\left(\mu_B - \mu_G\right)^2$	M1 M1 M1 A1 M1 A1 A1 **[7]**	 Answer given
1	(iv)		[Central Limit Theorem] Approx dist of $\overline{X}$ is $\mathrm{N}\left(\frac{1}{2}\mu_B + \frac{1}{2}\mu_G,\ \frac{1}{2n}\left(\sigma^2 + \frac{1}{4}\left(\mu_B - \mu_G\right)^2\right)\right)$ B1 B1 B1 B1	B4 **[4]**	4 marks as shown
1	(v)		$\overline{X}_{st} = \frac{1}{2}\left(\overline{X}_B + \overline{X}_G\right) \qquad \mathrm{Var}\left(\overline{X}_{either}\right) = \frac{\sigma^2}{n}$ $\therefore \mathrm{E}\left(\overline{X}_{st}\right) = \frac{1}{2}\left(\mu_B + \mu_G\right)$ and $\mathrm{Var}\left(\overline{X}_{st}\right) = \frac{1}{4}\left(\frac{\sigma^2}{n} + \frac{\sigma^2}{n}\right) = \frac{\sigma^2}{2n}$	M1M1 B1 B1 **[4]**	

Question			Answer	Marks	Guidance
1	(vi)		$E(\bar{X}) = E(\bar{X}_{st}) = \frac{1}{2}(\mu_B + \mu_G) = E(X)$	E1	
			ie they are unbiased.	E1	
			Clearly $\text{Var}(\bar{X}) > \text{Var}(\bar{X}_{st})$,	M1	for any attempt to compare variances Candidates are not required to note that the variances are equal in the case $\mu_B = \mu_G$.
				M1	for deduction that $\text{Var}(\bar{X}) > \text{Var}(\bar{X}_{st})$ [FT c's variances]
			$\therefore\ \bar{X}_{st}$ is the more efficient.	E1	More efficient
				[5]	
2	(i)		Mean of $X = 3.5$ (immediate by symmetry)	B1	
			$\text{E}(X^2) = \frac{1}{6}(1+4+\ldots+36) = \frac{91}{6}$	M1	
			$\therefore \text{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$	A1	Answer given
				[3]	
2	(ii)		$\text{G}(t) = \text{E}(t^X) = (t^1.\frac{1}{6}) + (t^2.\frac{1}{6}) + \ldots + (t^6.\frac{1}{6})$	M1	
			$= \frac{1}{6}(t + t^2 + \ldots + t^6) = \frac{t(1-t^6)}{6(1-t)}$	A1	Answer given
				[2]	
2	(iii)		[P(N = 0) = ½, P(N = 1) = (½)(½),] P($N = r$) = $(½)^r$.(½)	B1 **[1]**	answer given; must be convincing
2	(iv)		$\text{H}(t) = \text{E}(t^N) = (t^0.\frac{1}{2}) + (t^1.\frac{1}{4}) + (t^2.\frac{1}{8}) + \ldots$	M1	
			$= \frac{\frac{1}{2}}{1-\frac{t}{2}} = \frac{1}{2-t} = (2-t)^{-1}$	A1	Answer given

Question			Answer	Marks	Guidance
2	(v)		Mean = H'(1), variance = H''(1) + mean – mean^2. $H'(t) = (-1)(2-t)^{-2}(-1) = (2-t)^{-2}$ $\therefore$ mean = 1 $H''(t) = (-2)(2-t)^{-3}(-1) = 2(2-t)^{-3}$ $\therefore$ variance = 2 + 1 – 1 = 2	M1 A1 M1 A1 **[4]**	for <u>use</u> of 1st derivative for <u>use</u> of 2nd derivative For variance
2	(vi)		K(t) = H{G(t)} = {2 – G(t)}$^{-1}$ $= \left(2 - \frac{t(1-t^6)}{6(1-t)}\right)^{-1} = \left(\frac{12(1-t) - t(1-t)(1+t+t^2+...+t^5)}{6(1-t)}\right)^{-1}$ $= \left(\frac{12 - t - t^2 - t^3 - ... - t^6}{6}\right)^{-1} = 6(12 - t - t^2 - ... - t^6)^{-1}$	M1 M1 M1 A1 **[4]**	 inserting G(t) use of hint given Answer given
2	(vii)		$K'(t) = 6(12 - t - t^2 - ... - t^6)^{-2}(1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5)$ $K''(t) = 12(12 - t - t^2 - ... - t^6)^{-3}(1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5)^2$ $+ \ 6(12 - t - t^2 - ... - t^6)^{-2}(2 + 6t + 12t^2 + 20t^3 + 30t^4)$ $\therefore$ mean = K'(1) = 6(12 – 6)$^{-2}$(21) = 21/6 = 7/2 $\therefore$ K''(1) = (12 × 6^{-3} × 21^2) + (6 × 6^{-2} × 70) = (49/2) + (70/6) $\therefore$ variance = $\frac{49}{2} + \frac{70}{6} + \frac{7}{2} - \frac{49}{4} = \frac{294 + 140 + 42 - 147}{12} = \frac{329}{12}$	M1 M1 M1 A1 A1 A1 **[6]**	reasonable attempt to differentiate K(t) reasonable attempt at 2nd derivative for <u>use</u> of derivatives Substitution shown Ft c's K'(1) and/or K''(1) provided variance positive Exact.

Question			Answer	Marks	Guidance
2	(viii)		We have: $\mu_X = 7/2$ $\sigma_X^{\ 2} = 35/12$ $\mu_N = 1$ $\sigma_N^{\ 2} = 2$ $\sigma_Q^{\ 2} = 329/12$	M1	for correct use of candidate's values for means and variances
			Inserting in the quoted formula gives $\left[2\times\left(\frac{7}{2}\right)^2\right]+\left[1\times\frac{35}{12}\right] = \frac{294+35}{12} = \frac{329}{12}$ as required.	A1	answer honestly obtained (common denominator shown). A0 if different from (vii)
				[2]	
3	(i)		H_0: population medians are equal	B1	[*Note: "population" must be explicit*]
			H_1: population median for A < population median for B	B1	1) Explicit statement re shapes of distributions. (eg that they are the same shape) is not required. 2) More formal statements of hypotheses gain both marks [eg cdfs are F(x) and F(x – Δ), H_0 is Δ = 0 etc).]
			Wilcoxon rank sum test (or Mann-Whitney form of test)		
			Ranks are: A 1 2 4 5 9 11	M1	Combined ranking
			B 3 6 7 8 10 12 13 14	A1	Correct [allow up to 2 errors; FT provided M1 earned]
			W = 1 + 2 + 4 + 5 + 9 + 11 = 32 [or 0 + 0 + 1 + 1 + 4 + 5 = 11 if M-W used]	B1	
			Refer to $W_{6,8}$ [or $MW_{6,8}$] tables	M1	No FT if wrong
			Lower 5% critical point is 31 [or 10 if M-W used]	A1	No FT if wrong
			Result is not significant	A1	
			Seems median yields may be assumed equal	A1	
				[9]	

Question			Answer	Marks	Guidance
3	(ii)		H_0: population means are equal H_1: population mean for A < population mean for B For A: $\bar{x} = 11.4$, $s_{n-1}^2 = 1.912$ [$s_{n-1} = 1.38275$] For B: $\bar{y} = 12.575$, $s_{n-1}^2 = 1.051$ [$s_{n-1} = 1.025$] Pooled $s^2 = \frac{(5\times1.912)+(7\times1.051)}{12} = \frac{16.915}{12} = 1.4096$ Test statistic $= \frac{11.4-12.575}{\sqrt{1.4096}\sqrt{\frac{1}{6}+\frac{1}{8}}} = \frac{-1.175}{0.6412} = -1.83(25)$ Refer to t_{12} Lower single-tailed 5% critical point is –1.782 Significant Seems mean yield for A is less than that for B	B1 B1 B1 M1 A1 M1 A1 M1 A1 A1 A1 **[11]**	 *"population" must be explicit, either in words or by notation* For all. Use of s_n scores B0 for any reasonable attempt at pooling (but *not* if s_n^2 used) If correct Ft if incorrect No FT if wrong No FT if wrong *must compare –1.83 with –1.782 unless it is clear and explicit that absolute values are being used*
3	(iii)		t test is "more sensitive" if Normality is correct. Non-rejection of Normality supports t. But Wilcoxon is more reliable if not Normal – and we do not have *proof* of Normality.	E1 E1 E1 E1 **[4]**	
4	(i)		Latin square 4 × 4 layout, with rows clearly representing casting techniques and columns clearly representing chemical compositions [*or vice versa*] Labels each appearing exactly once in each row and in each column representing manufacturers.	B1 B2,1,0 B1 B1 **[5]**	 B2 if completely correct, B1 if one point omitted for correct structure within the square

<table>
<tr><th colspan="3">Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td>4</td><td>(ii)</td><td></td><td>Totals are 440.6 453.5 458.2 459.4 all from samples of size 4
Grand total 1811.7 "Correction factor" CF = $1811.7^2/16$
= 205141.06
Total SS = 205202.57 – CF = 61.5144
Between manufacturers SS =
$\frac{440.6^2}{4}+\frac{453.5^2}{4}+\frac{458.2^2}{4}+\frac{459.4^2}{4}-\text{CF}$
= 205196.55 – CF = 55.4969
Residual SS (by subtraction) = 61.5144 – 55.4969 = 6.0175

Source of variation | SS | df | MS | MS ratio
Between treatments | 55.4969 | 3 | 18.4989 | 36.89
Residual | 6.0175 | 12 | 0.5014(6)
Total | 61.5144 | 15

Refer MS ratio to $F_{3,12}$.
quotation of an extreme point (not just the 5%point); eg 1% point, 5.95, or 0.1% point, 10.80.
Must be some reference to very highly significant
and very strong/overwhelming evidence that "the manufacturers are not all the same".</td><td>

M1

M1
A1

B1
B1
M1
M1
A1

M1
A1

A1
A1

[12]</td><td>

for attempt to form three sums of squares.

for correct method for any two
if each calculated SS is correct.

Between treatments df
Residual df
Method for calculating either mean square
MS ratio
cao at least 3sf, condone up to 6 sf

No FT if wrong</td></tr>
<tr><td>4</td><td>(iii)</td><td></td><td>$y_{ij}=\mu+\alpha_i+e_{ij}$
μ = population …
…grand mean for whole exp't
α_i = population mean amount by which the ith treatment differs from μ

e_{ij} ~ ind N [*] (accept "uncorrelated") (0 [*] , σ^2 [*])</td><td>B2
B1
B1
B1

B2
[7]</td><td>B1 if any two RHS terms correct
"Population"; award here <u>or</u> for α_i

e_{ij} is experimental error – does not need to be stated explicitly here, is subsumed in error assumptions below.

if all three * components correct, B1 if any two correct.</td></tr>
</table>

Question			Answer	Marks	Guidance
1	(i)		$\mathrm{P}(X = x) = \dfrac{\mathrm{e}^{-\theta}\theta^x}{x!}$		
			$\mathrm{L} = \dfrac{\mathrm{e}^{-\theta}\theta^{x_1}}{x_1!} \cdot \ldots \cdot \dfrac{\mathrm{e}^{-\theta}\theta^{x_n}}{x_n!} = \dfrac{\mathrm{e}^{-n\theta}\theta^{\Sigma x_i}}{x_1!x_2!\ldots x_n!}$	M1 A1	M1 for general product form. A1 (a.e.f.) for answer.
			$\ln \mathrm{L} = -n\theta + \Sigma x_i \ln\theta - \Sigma \ln x_i!$	M1 A1	M1 is for taking logs (base e). Allow (±) constant instead of last term.
			$\dfrac{\mathrm{d}\ln \mathrm{L}}{\mathrm{d}\theta} = -n + \dfrac{\Sigma x_i}{\theta}$	M1 A1	M1 for differentiating, A1 for answer.
			$= 0$ for ML Est $\hat{\theta}$.	M1	Any or all of the four M1 marks down to here can be awarded in part (iv) if not awarded here.
			$\therefore n\hat{\theta} = \Sigma x_i$, i.e. $\hat{\theta} = \bar{x}$.	A1	
			Confirmation that this is a maximum:	M1	
			$\dfrac{\mathrm{d}^2 \ln \mathrm{L}}{\mathrm{d}\theta^2} = -\dfrac{\Sigma x_i}{\theta^2} < 0$.	A1 **[10]**	
1	(ii)		$\mathrm{P}(X = 0) = \mathrm{e}^{-\theta}$.	M1	
			ML Est of $\mathrm{e}^{-\theta} = \mathrm{e}^{-\hat{\theta}}$, i.e. the estimate is $\mathrm{e}^{-\bar{x}}$.	M1 A1 **[3]**	M1 for "invariance property", not necessarily named.
1	(iii)		We have estimate of $\mathrm{P}(X = 0) = \mathrm{e}^{-5} = 0.0067$,	M1	
			so we might reasonably expect around $1000e^{-5} \approx 6.7$ cases of zero in a sample of size 1000	E1	Sensible use of $n = 1000$ and $\bar{x} = 5$.
			– finding no such cases seems suspicious.	E1 **[3]**	

Question			Answer	Marks	Guidance
1	(iv)		X has Poisson distribution "scaled up" so that , therefore ,	M1	M1 for this idea, however expressed.
			where $1 = k\sum_{x=1}^{\infty}\frac{e^{-\theta}\theta^{x}}{x!} = k\left\{\sum_{x=0}^{\infty}\frac{e^{-\theta}\theta^{x}}{x!} - e^{-\theta}\right\} = k\left\{1-e^{-\theta}\right\}$.	M1	M1 for sum from 0 to ∞ minus value for $x = 0$.
			$\therefore k = \frac{1}{1-e^{-\theta}}$.	M1	
			$\therefore P(X = x) = \frac{1}{1-e^{-\theta}}.\frac{e^{-\theta}\theta^{x}}{x!} = \frac{\theta^{x}}{(e^{\theta}-1)x!}$ [for x = 1, 2, ...].	A1	**Beware printed answer.**
			$\therefore L = \frac{\theta^{\Sigma x_i}}{(e^{\theta}-1)^{n}\, x_1!x_2!\ldots x_n!}$	A1	
			$\therefore \ln L = \Sigma x_i \ln\theta - n\ln(e^{\theta}-1) - \Sigma \ln x_i!$	A1	
			$\therefore \frac{d\ln L}{d\theta} = \frac{\Sigma x_i}{\theta} - \frac{ne^{\theta}}{e^{\theta}-1}$,	A1	
			and on setting this equal to zero we get that $\hat{\theta}$ satisfies $\frac{\theta e^{\theta}}{e^{\theta}-1} = \frac{\Sigma x_i}{n} = \bar{x}$.	A1	**Beware printed answer.**
				[8]	
2	(i)	(*A*)	$\mu = 0$.	B1	
		(*B*)	$E(X^2) = 8/3$.	B1	
		(*C*)	$Var(X) = 8/3$.	B1	
				[3]	
2	(ii)		Mgf of X is $M_X(\theta) = E(e^{\theta X})$		
			$= \left(e^{-2\theta}.\frac{1}{3}\right) + \left(e^{0}.\frac{1}{3}\right) + \left(e^{2\theta}.\frac{1}{3}\right)$	M1	
			$= \frac{1}{3}\left(1 + e^{2\theta} + e^{-2\theta}\right)$.	A1	Any equivalent form.
				[2]	

Question			Answer	Marks	Guidance
2	(iii)		General results:		
			[Convolution theorem] Mgf of sum of independent random variables = product of their mgfs.	B1 M1	B1 for explicit mention of "independent". Note M1 mark required for f.t. to A marks below.
			[Linear transformation result] $M_{aX+b}(\theta) = e^{b\theta} M_X(a\theta)$.	M1	Allow implicit $b = 0$. Note M1 mark required for f.t. to A marks below.
			$M_{\Sigma X}(\theta) = \left\{\frac{1}{3}\left(1 + e^{2\theta} + e^{-2\theta}\right)\right\}^n$	A1	
			$M_{\bar{X}}(\theta) = \left\{\frac{1}{3}\left(1 + e^{2\frac{\theta}{n}} + e^{-2\frac{\theta}{n}}\right)\right\}^n$	A1	
			$Z = \frac{\bar{X}\sqrt{n}}{\sigma} = \frac{\sqrt{n}\bar{X}\sqrt{3}}{2\sqrt{2}}$	B1	Might be implicit in what follows.
			$M_Z(\theta) = \left\{\frac{1}{3}\left(1 + e^{\frac{2}{n}\frac{\sqrt{3n}}{2\sqrt{2}}\theta} + e^{-\frac{2}{n}\frac{\sqrt{3n}}{2\sqrt{2}}\theta}\right)\right\}^n$	A1	
			$= \left\{\frac{1}{3}\left(1 + e^{\frac{\theta\sqrt{3}}{\sqrt{2n}}} + e^{-\frac{\theta\sqrt{3}}{\sqrt{2n}}}\right)\right\}^n$.	A1	**Beware printed answer.**
				[8]	

Question			Answer	Marks	Guidance
2	(iv)		$M_Z(\theta) = \{ \frac{1}{3}(1$	M1	M1 for reasonable attempt to expand exponentials.
			$+1+\frac{\theta\sqrt{3}}{\sqrt{2n}}+\frac{3\theta^2}{\sqrt{2}n.2!}+\text{terms in } n^{-3/2}, n^{-2}, \ldots$	A1	A1 for this line.
			$+1-\frac{\theta\sqrt{3}}{\sqrt{2n}}+\frac{3\theta^2}{\sqrt{2}n.2!}+\text{terms in } n^{-3/2}, n^{-2}, \ldots)\}^n$	A1	A1 for this line.
			cancel $\surd$	M1	M1 for cancelling first order terms, may be implicit.
			neglect	M1	M1 for neglecting higher order terms in n^{-1}, MUST be explicit.
			$\approx \left\{\frac{1}{3}\left(3+\frac{3\theta^2}{2n}\right)\right\}^n$	A1	
			$= \left(1+\frac{\theta^2}{2n}\right)^n.$	A1	**Beware printed answer.**
				[7]	
2	(v)		Limit is $e^{\theta^2/2}$. This is mgf of N(0, 1).	M1 M1	M1 for limit, M1 for recognising mgf.
			Mgfs are unique (even in this limiting process).	B1	Bracketed phrase is not needed to earn the mark.
			So (approximately) distribution of Z is N(0, 1).	A1	Bracketed word is not needed to earn the mark.
				[4]	
3	(i)		Type I error: rejecting null hypothesis	B1	Allow B1 out of 2 for P(...).
			when it is true	B1	
			Type II error: accepting null hypothesis	B1	Allow B1 out of 2 for P(...).
			when it is false	B1	
			OC: P(accepting null hypothesis	B1	P(Type II error | the true value of the parameter) scores B1+B1.
			as a function of the parameter under investigation)	B1	
			Power: P(rejecting null hypothesis	B1	P(Type I error | the true value of the parameter) scores B1+B1.
			as a function of the parameter under investigation)	B1	
				[8]	

Question			Answer	Marks	Guidance
3	(ii)		Correct "spike" shape, or point shown. Location of spike or point at θ_0 correct and correctly labelled. Height of spike correct and correctly labelled as 1.	G1 G1 G1 **[3]**	
3	(iii)		$X \sim N(\mu, 9)$. $n = 25$. H_0: $\mu = 0$. H_1: $\mu \neq 0$. Accept H_0 if $-1 < \bar{x} < 1$. P(Type I error) = $P\left(\left\|\bar{X}\right\| > 1 \;\middle\|\; \bar{X} \sim N\left(0, \frac{9}{25}\right)\right)$ $= P\left(\left\|N(0,1)\right\| > \frac{1-0}{3/5}\right) = 2 \times 0.0478 = 0.0956.$ P(Type II error when $\mu = 0.5$) $= P\left(-1 < \bar{X} < 1 \;\middle\|\; \bar{X} \sim N\left(0.5, \frac{9}{25}\right)\right)$ $= P\left(\frac{-1.5}{3/5} < N(0,1) < \frac{0.5}{3/5}\right) = P(-2.5 < N(0,1) < 0.8333)$ $= 0.7976 - 0.0062 = 0.7914.$ This is high. We are trying to detect only a small departure from H_0, the "error" (σ^2) being comparatively large.	 M1 M1 A1 M1 M1 M1 A1 E1 E1 **[9]**	 M1 for use of $\left\|\bar{X}\right\| > 1$, M1 for distribution of $\bar{X}$. Either or both marks might be implicit in what follows. Accept a.w.r.t. 0.096. As for the two M1 marks for the Type I error. Standardising with correct end-points.
3	(iv)		Correct shape – must be symmetrical, and reasonable approximation to Normal pdf. "Centre" at 0 and clearly labelled. Height at 0 distinctly less than 1 by about 10% (ft candidate's P(Type I error)). Some indication that height at 0.5 is about 0.8 (ft candidate's P(Type II error)).	G1 G1 G1 G1 **[4]**	

Question			Answer	Marks	Guidance
4	(i)		Randomisation is mainly to guard against possible sources of bias, which may be due to subjective allocation of treatments to units or unsuspected. Replication enables an estimate of experimental error to be made.	B1 B1 B1 B1 B1 B1 **[6]**	Award up to B3 marks. These include B1 for idea of "sources of bias" and another B1 for either idea of " unsuspected" or “subjective”. Award up to B3 marks. These include B1 for *some* concept of replication addressing experimental error and another B1 for the explicit idea that it enables this error to be estimated. The third B1 should be awarded for the general quality of the response, especially the lack of extraneous material in it Alternative suggestions offered by candidates may be rewarded *provided they are statistically sound*. Suggestions that are limited to particular designs (eg for the advantages of randomisation within a randomised blocks design) may be rewarded if statistically sound, but for at most 2 of the B3 marks in each set.
4	(ii)		μ is the population mean for the entire experiment. α_i is the population amount by which the mean for the ith treatment differs from μ. e_{ij} ~ ind N (0, σ^2)	B1 B1 B1 B1 B1 B1 B1 **[7]**	B1 for explicit mention of "population", B1 for idea of mean for whole experiment. B1 for explicit mention of "population", B1 for idea of difference of means. For "ind N"; allow "uncorrelated". For mean 0. For variance σ^2 [i.e. that the variance is constant].

<table>
<tr><th colspan="3">Question</th><th>Answer</th><th>Marks</th><th>Guidance</th></tr>
<tr><td>4</td><td>(iii)</td><td></td><td>Null hypothesis: $\alpha_1 = \alpha_2 = ... = \alpha_k$ (= 0) Alternative hypothesis: not all α_i are equal Note that alternative hypothesis is NOT that all the α_i are different – B0 for alternative hypothesis if this is stated. | Source of variation	Sum of squares	d.f.	Mean squares	MS ratio
Between fertilisers	219.2	4	54.8	2.7
Residual	304.5	15	20.3	
Total	523.7	19		| Refer 2.7 to $F_{4,15}$. Upper 5% point is 3.06. Not significant. Seems mean effects of fertilisers are all the same.</td><td>B1 B1 B1 B1 M1 M1 A1 M1 A1 A1 E1 **[11]**</td><td>No need for definition of k as the number of treatments, and accept simply $\alpha_1 = \alpha_2 =...$ with no explicit upper end to the sequence. Accept hypotheses stated verbally provided it is clear that *population* parameters (means) are being referred to. B1 for *each* d.f. (4 and 15). For *method* of mean squares. For *method* of mean square ratio. A1 c.a.o. for 2.7 (2.6695). f.t. from here provided all M marks earned. No f.t. if wrong but allow M1 for F with candidate's df if both positive and totalling 19. cao. No f.t. if wrong (or if not quoted). Verbal conclusion in context, and not "too assertive".</td></tr>
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