Solutionbank S3
Edexcel AS and A Level Modular Mathematics

Combinations of random variables
Exercise A, Question 1

Question:

Given the random variables $X \sim N(80, 3^2)$ and $Y \sim N(50, 2^2)$ where X and Y are independent find the distribution of W where:

a $W = X + Y$,

b $W = X - Y$.

Solution:

a
$$\begin{aligned} E(W) &= E(X) + E(Y) \\ &= 80 + 50 \\ &= 130 \end{aligned}$$

$$\begin{aligned} \text{Var}(W) &= \text{Var}(X) + \text{Var}(Y) \\ &= 9 + 4 \\ &= 13 \end{aligned}$$

$$W \sim N(130, 13)$$

b
$$\begin{aligned} E(W) &= E(X) - E(Y) \\ &= 80 - 50 \\ &= 30 \end{aligned}$$

$$\begin{aligned} \text{Var}(W) &= \text{Var}(X) + \text{Var}(Y) \\ &= 9 + 4 \\ &= 13 \end{aligned}$$

$$W \sim N(30, 13)$$

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Combinations of random variables
Exercise A, Question 2

Question:

Given the random variables $X \sim N(45, 6)$, $Y \sim N(54, 4)$ and $W \sim N(49, 8)$ where X, Y and W are independent, find the distribution of R where $R = X + Y + W$.

Solution:

$$\begin{aligned}
E(R) &= E(X) + E(Y) + E(W) \\
&= 45 + 54 + 49 \\
&= 148 \\
\text{Var}(R) &= \text{Var}(X) + \text{Var}(Y) + \text{Var}(W) \\
&= 6 + 4 + 8 \\
&= 18 \\
R &\sim N(148, 18)
\end{aligned}$$

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Combinations of random variables
Exercise A, Question 3

Question:

X_1 and X_2 are independent normal random variables. $X_1 \sim \mathrm{N}(60, 25)$ and $X_2 \sim \mathrm{N}(50,16)$. Find the distribution of T where:

a $T = 3X_1$,

b $T = 7X_2$,

c $T = 3X_1 + 7X_2$,

d $T = X_1 - 2X_2$.

Solution:

a
$$\begin{aligned}E(T) &= 3E(X_1)\\ &= 3\times 60\\ &= 180\\ \text{Var}(T) &= 9\,\text{Var}(X_1)\\ &= 9\times 25\\ &= 225\end{aligned}$$
$T \sim N(180, 225)$ or $N(180, 15^2)$

b
$$\begin{aligned}E(T) &= 7E(X_2)\\ &= 7\times 50\\ &= 350\\ \text{Var}(T) &= 49\,\text{Var}(X_2)\\ &= 49\times 16\\ &= 784\end{aligned}$$
$T \sim N(350, 784)$ or $N(350, 28^2)$

c
$$\begin{aligned}E(T) &= E(3X_1) + E(7X_2)\\ &= 180 + 350\\ &= 530\\ \text{Var}(T) &= \text{Var}(3X_1) + \text{Var}(7X_2)\\ &= 225 + 784\\ &= 1009\end{aligned}$$
$T \sim N(530, 1009)$

d
$$\begin{aligned}E(T) &= E(X_1) - 2E(X_2)\\ &= 60 - 2\times 50\\ &= -40\\ \text{Var}(T) &= \text{Var}(X_1) + 4\text{Var}(X_2)\\ &= 25 + 4\times 16\\ &= 89\end{aligned}$$
$T \sim N(-40, 89)$

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Combinations of random variables
Exercise A, Question 4

Question:

Y_1, Y_2 and Y_3 are independent normal random variables.
$Y_1 \sim N(8,2), Y_2 \sim N(12,3)$ and $Y_3 \sim N(15,4)$. Find the distribution of A where:

a $A = Y_1 + Y_2 + Y_3$,

b $A = Y_3 - Y_1$,

c $A = Y_1 - Y_2 + 3Y_3$,

d $A = 3Y_1 + 4Y_3$,

e $A = 2Y_1 - Y_2 + Y_3$.

Solution:

a $\text{E}(A) = \text{E}(Y_1) + \text{E}(Y_2) + \text{E}(Y_3)$

$= 8 + 12 + 15$

$= 35$

$\text{Var}(A) = \text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3)$

$= 2 + 3 + 4$

$= 9$

$A \sim \text{N}(35, 9)$ or $\text{N}(35, 3^2)$

b $\text{E}(A) = \text{E}(Y_3) - \text{E}(Y_1)$

$= 15 - 8$

$= 7$

$\text{Var}(A) = \text{Var}(Y_3) + \text{Var}(Y_1)$

$= 4 + 2$

$= 6$

$A \sim \text{N}(7, 6)$

c $\text{E}(A) = \text{E}(Y_1) - \text{E}(Y_2) + 3\text{E}(Y_3)$

$= 8 - 12 + 3 \times 15$

$= 41$

$\text{Var}(A) = \text{Var}(Y_1) + \text{Var}(Y_2) + 9\text{Var}(Y_3)$

$= 2 + 3 + 9 \times 4$

$= 41$

$A \sim \text{N}(41, 41)$

d $\text{E}(A) = 3\text{E}(Y_1) + 4\text{E}(Y_3)$

$= 3 \times 8 + 4 \times 15$

$= 84$

$\text{Var}(A) = 9\text{Var}(Y_1) + 16\text{Var}(Y_3)$

$= 9 \times 2 + 16 \times 4$

$= 82$

$A \sim \text{N}(84, 82)$

e $\text{E}(A) = 2\text{E}(Y_1) - \text{E}(Y_2) + 3\text{E}(Y_3)$

$= 2 \times 8 - 12 + 15$

$= 19$

$\text{Var}(A) = 4\text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3)$

$= 4 \times 2 + 3 + 4$

$= 15$

$A \sim \text{N}(19, 15)$

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Combinations of random variables
Exercise A, Question 5

Question:

A, B and C are independent normal random variables.
$A \sim N(50,6), B \sim N(60,8)$ and $C \sim N(80,10)$. Find:

a $P(A+B<115)$,

b $P(A+B+C>198)$,

c $P(B+C<138)$,

d $P(2A+B-C<70)$,

e $P(A+3B-C>140)$,

f $P(105<(A+B)<116)$.

Solution:

a $A + B \sim N(50 + 60, 6 + 8) = N(110, 14)$

$$P(A + B < 115) = P\left(z < \frac{115 - 110}{\sqrt{14}}\right)$$
$$= P(z < 1.34)$$
$$= 0.9099\ (0.9093)$$

Answers which round to (awrt) 0.91

b $A + B + C \sim N(50 + 60 + 80, 6 + 8 + 10) = N(190, 24)$

$$P(A + B + C > 198) = P\left(z > \frac{198 - 190}{\sqrt{24}}\right)$$
$$= P(z < 1.63)$$
$$= 1 - 0.9484$$
$$= 0.0516\ \ (0.0512)$$

c $B + C \sim N(60 + 80, 8 + 10) = N(140, 18)$

$$P(B + C < 138) = P\left(z < \frac{138 - 140}{\sqrt{18}}\right)$$
$$= P(z < -0.47)$$
$$= 1 - 0.6808$$
$$= 0.3192\ (0.3186)$$

Awrt 0.319

d $2A + B - C \sim N(2\times 50 + 60 - 80, 4\times 6 + 8 + 10) = N(80, 42)$

$$P(2A + B - C < 70) = P\left(z < \frac{70 - 80}{\sqrt{42}}\right)$$
$$= P(z < -1.54)$$
$$= 1 - 0.9382$$
$$= 0.0618\ \ (0.0614)$$

e $A + 3B - C \sim N(50 + 3\times 60 - 80, 6 + 9 \times 8 + 10) = N(150, 88)$

$$P(A + 3B - C > 140) = P\left(z > \frac{140 - 150}{\sqrt{88}}\right)$$
$$= P(z > -1.07)$$
$$= 0.8577\ (0.8568)$$

Awrt 0.858 (0.857)

f $A + B \sim N(50 + 60, 6 + 8) = N(110, 14)$

$$P(105 < A + B < 116) = P\left(\frac{105 - 110}{\sqrt{14}} < z < \frac{116 - 110}{\sqrt{14}}\right)$$
$$= P(-1.34 < z < 1.60)$$
$$= 0.9452 - (1 - 0.9099)$$
$$= 0.8551\ (0.8549)$$

Awrt 0.855

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Combinations of random variables
Exercise A, Question 6

Question:

Given the random variables $X \sim N(20, 5)$ and $Y \sim N(10, 4)$ where X and Y are independent, find

a $E(X - Y)$,

b $Var(X - Y)$,

c $P(13 < X - Y < 16)$. ***E***

Solution:

a $E(X - Y) = 20 - 10 = 10$

b $Var(X - Y) = 5 + 4 = 9$

c $X - Y \sim N(10, 9)$

$$\begin{aligned} P(13 < X - Y < 16) &= P(X - Y < 16) - P(X - Y < 13) \\ &= P(z < 2) - P(z < 1) \\ &= 0.9772 - 0.8413 \\ &= 0.1359 \\ &\text{Awrt } 0.136 \end{aligned}$$

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Combinations of random variables
Exercise A, Question 7

Question:

The random variable R is defined as $R = X + 4Y$ where $X \sim \text{N}(8, 2^2)$, $Y \sim \text{N}(14, 3^2)$ and X and Y are independent.

Find

a $\text{E}(R)$,

b $\text{Var}(R)$,

c $\text{P}(R < 41)$.

The random variables Y_1, Y_2 and Y_3 are independent and each has the same distribution as Y. The random variable S is defined as

$$S = \sum_{i=1}^{3} Y_i - \frac{1}{2}X.$$

d Find Var (S). ***E***

Solution:

a $\text{E}(R) = \text{E}(X) + 4\text{E}(Y) = 8 + (4 \times 14) = 64$

b $\text{Var}\ (R) = \text{Var}\ (X) + 16\ \text{Var}\ (Y) = 2^2 + (16 \times 3^2)$
$= 148$

c $\text{P}(R < 41) = \text{P}\left(Z < \frac{41 - 64}{\sqrt{148}}\right) = \text{P}(Z < -1.89)$
$= 0.0294\ (\ 0.0293)$

d $S = Y_1 + Y_2 + Y_3 - 0.5X$
$\text{Var}\ (S) = 3\ \text{Var}\ (Y) + (\frac{1}{2})^2\ \text{Var}\ (X)$
$= 27 + 1$
$= 28$

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Combinations of random variables
Exercise A, Question 8

Question:

A factory makes steel rods and steel tubes. The diameter of a steel rod is normally distributed with mean 3.55 cm and standard deviation 0.02 cm. The internal diameter of a steel tube is normally distributed with mean 3.60 cm and standard deviation 0.02 cm.
A rod and a tube are selected at random. Find the probability that the rod cannot pass through the tube. ***E***

Solution:

$T \sim N(3.60, 0.02^2)$ $R \sim N(3.55, 0.02^2)$
$P(T < R) = P(T - R < 0)$
$E(T - R) = 3.60 - 3.55 = 0.05$
$Var(T - R) = 0.02^2 + 0.02^2 = 0.0008$

$$\begin{aligned} P(T - R < 0) &= P\left(z < \frac{0 - 0.05}{\sqrt{0.0008}}\right) \\ &= P(z < -1.77) \\ &= 1 - 0.9616 \\ &= 0.0384\ (0.0385) \end{aligned}$$

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Combinations of random variables
Exercise A, Question 9

Question:

The weight of a randomly selected tin of jam is normally distributed with a mean weight of 1 kg and a standard deviation of 12 g. The tins are packed in boxes of 6 and the weight of the box is normally distributed with mean weight 250 g and standard deviation 10 g. Find the probability that a randomly chosen box of 6 tins will weigh less than 6.2 kg. ***E***

Solution:

$T \sim \text{N}(1000, 12^2)$ $B \sim \text{N}(250, 10^2)$
$Y = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + B_1$
$\text{E}(Y) = 6 \times 1000 + 250 = 6250$
$\text{Var}(Y) = 6 \times 12^2 + 10^2 = 964$

$$\begin{aligned}\text{P}(Y < 6200) &= \text{P}\left(z < \frac{6200 - 6250}{\sqrt{964}}\right) \\ &= \text{P}(z < -1.61) \\ &= 1 - 0.9463 \\ &= 0.0537\end{aligned}$$

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Combinations of random variables
Exercise A, Question 10

Question:

The thickness of paperback books can be modelled as a normal random variable with mean 2.1 cm and variance $0.39\ \text{cm}^2$. The thickness of hardback books can be modelled as a normal random variable with mean 4.0 cm and variance $1.56\ \text{cm}^2$. A small bookshelf is 30 cm long.

a Find the probability that a random sample of
- **i** 15 paperback books can be placed side-by-side on the bookshelf,
- **ii** 5 hardback and 5 paperback books can be placed side-by-side on the bookshelf.

b Find the shortest length of bookshelf needed so that there is at least a 99% chance that it will hold a random sample of 15 paperback books. ***E***

Solution:

$P \sim N(2.1, 0.39)$ $H \sim N(4.0, 1.56)$

a **i** $Y = P_1 + P_2 + P_3 + \ldots + P_{15}$

$E(Y) = 15 \times 2.1 = 31.5$
$Var(Y) = 15 \times 0.39 = 5.85$

$$P(Y < 30) = P\left(z < \frac{30 - 31.5}{\sqrt{5.85}}\right)$$
$$= P(z < -0.62)$$
$$= 1 - 0.7324$$
$$= 0.2676$$

Awrt 0.268

ii $Y = P_1 + P_2 + \ldots + P_5 + H_1 + H_2 + \ldots + H_5$

$E(Y) = 5 \times 2.1 + 5 \times 4.0 = 30.5$
$Var(Y) = 5 \times 0.39 + 5 \times 1.56 = 9.75$

$$P(Y < 30) = P\left(z < \frac{30 - 30.5}{\sqrt{9.75}}\right)$$
$$= P(z < -0.16)$$
$$= 1 - 0.5636$$
$$= 0.4364$$

Awrt 0.436

b $Y = P_1 + P_2 + P_3 + \ldots + P_{15}$

$E(Y) = 15 \times 2.1 = 31.5$
$Var(Y) = 15 \times 0.39 = 5.85$
$P(Y < x) > 0.99$

$$P\left(z < \frac{x - 31.5}{\sqrt{5.85}}\right) > 0.99$$

$$\frac{x - 31.5}{\sqrt{5.85}} > 2.3263$$

$x = 37.1$

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Combinations of random variables
Exercise A, Question 11

Question:

A sweet manufacturer produces two varieties of fruit sweet, Xtras and Yummies. The weights, X and Y in grams, of randomly selected Xtras and Yummies are such that

$X \sim \mathrm{N}(30,25)$ and $Y \sim \mathrm{N}(32,16)$.

a Find the probability that the weight of two randomly selected Yummies will differ by more than 5 g.

One sweet of each variety is selected at random.

b Find the probability that the Yummy sweet weighs more than the Xtra.

A packet contains 6 Xtras and 4 Yummies.

c Find the probability that the average weight of the sweets in the packet lies between 28 g and 33 g. ***E***

Solution:

a $W = Y_1 - Y_2$

$\mathrm{E}(W) = \mathrm{E}(Y_1) - \mathrm{E}(Y_2) = 0$

$\mathrm{Var}(W) = \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) = 32$

$\mathrm{P}(|W| > 5) = \mathrm{P}(W < -5) + \mathrm{P}(W > 5)$

$= \mathrm{P}\left(z < \dfrac{-5-0}{\sqrt{32}}\right) + \mathrm{P}\left(z > \dfrac{5-0}{\sqrt{32}}\right)$

$= \mathrm{P}(z < -0.88) + \mathrm{P}(z > 0.88)$

$= (1 - 0.8106) + (1 - 0.8106)$

$= 0.3788\ (0.3768)$

Awrt 0.379/0.377

b $W = Y_1 - X_1$

$\mathrm{E}(W) = \mathrm{E}(Y_1) - \mathrm{E}(X_1) = 2$

$\mathrm{Var}(W) = \mathrm{Var}(Y_1) + \mathrm{Var}(X_1) = 41$

$\mathrm{P}(W > 0) = \mathrm{P}\left(z > \dfrac{0-2}{\sqrt{41}}\right)$

$= \mathrm{P}(z > -0.31)$

$= 0.6217\ (0.6226)$

Awrt 0.622/0.623

c $W = X_1 + X_2 + \ldots + X_6 + Y_1 + Y_2 + \ldots + Y_4$

$\mathrm{E}(W) = 6\mathrm{E}(X) + 4\mathrm{E}(Y) = 308$

$\mathrm{Var}(W) = 6\mathrm{Var}(X) + 4\mathrm{Var}(Y) = 214$

Since the total weight of the 10 sweets must be between 280 g and 330 g

$\mathrm{P}(28 < \overline{W} < 33) = \mathrm{P}\left(\dfrac{280-308}{\sqrt{214}} < z < \dfrac{330-308}{\sqrt{214}}\right)$

$= \mathrm{P}(-1.91 < z < 1.50)$

$= 0.9332 - (1 - 0.9719)$

$= 0.9051\ (0.9059)$

Awrt 0.905/0.906

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Combinations of random variables
Exercise A, Question 12

Question:

If $X_1, X_2, \ldots, X_n$, are independent random variables, each with mean μ and variance σ^2, and the random variable Z is defined as $Z = X_1 + X_2 + \ldots + X_n$, show that $\mathrm{E}(Z) = n\mu$ and $\mathrm{Var}(Z) = n\sigma^2$.

A certain brand of biscuit is individually wrapped. The weight of a biscuit can be taken to be normally distributed with mean 75 g and standard deviation 5 g. The weight of an individual wrapping is normally distributed with mean 10 g and standard deviation 2 g. Six of these individually wrapped biscuits are then packed together. The weight of the packing material is a normal random variable with mean 40 g and standard deviation 3 g. Find, to 3 decimal places, the probability that the total weight of the packet lies between 535 g and 565 g. ***E***

Solution:

$$\begin{aligned}\mathrm{E}(Z) &= \mathrm{E}(X_1) + \mathrm{E}(X_2) + \ldots + \mathrm{E}(X_n)\\ &= \mu + \mu + \ldots + \mu\\ &= n\mu\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Z) &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)\\ &= \sigma^2 + \sigma^2 + \ldots + \sigma^2\\ &= n\sigma^2\end{aligned}$$

$B \sim \mathrm{N}(75, 5^2)$ $W \sim \mathrm{N}(10, 2^2)$ $X \sim \mathrm{N}(40, 3^2)$

$W = B_1 + B_2 + \ldots + B_6 + W_1 + W_2 + \ldots + W_6 + X$

$\mathrm{E}(W) = 6\mathrm{E}(B) + 6\mathrm{E}(W) + \mathrm{E}(X) = 550$

$\mathrm{Var}(W) = 6\mathrm{Var}(B) + 6\mathrm{Var}(W) + \mathrm{Var}(X) = 183$

$$\begin{aligned}\mathrm{P}(535 < W < 565) &= \mathrm{P}\left(\frac{535-550}{\sqrt{183}} < z < \frac{565-550}{\sqrt{183}}\right)\\ &= \mathrm{P}(-1.11 < z < 1.11)\\ &= 0.8665 - (1 - 0.8665)\\ &= 0.733\ (0.732)\end{aligned}$$