Exercise 4B

1 a Let X be the random discrete variable $X \sim \text{Po}(3)$ and let T denote the sum of the 10 sample observations, so $T \sim \text{Po}(10 \times 3)$, i.e. $T \sim \text{Po}(30)$

If the sample mean = 2.5, then $T = 10 \times 2.5 = 25$. So the probability that the sample mean is less than 25 is $\text{P}(T \leqslant 25)$

By calculation $\text{P}(T \leqslant 25) = 0.2084$ (4 d.p.)

b By the central limit theorem, $\bar{X} \approx\sim \text{N}\left(3, \frac{3}{10}\right)$, i.e. $X \approx\sim \text{N}(3, 0.3)$

Using a calculator, $\text{P}(\bar{X} \leqslant 2.5) = 0.1807$ (4 d.p.)

The two answers are not very close. This is because the estimate found in part **b** is not very accurate as the sample size is too small.

2 $X \sim \text{B}(10, 0.2)$

$\text{E}(X) = np = 10 \times 0.2 = 2$

$\text{Var}(X) = np(1-p) = 2 \times 0.8 = 1.6$

By the central limit theorem $\bar{X} \approx\sim \text{N}\left(2, \frac{1.6}{20}\right)$, i.e. $\bar{X} \approx\sim \text{N}(2, 0.8)$

$\text{P}(\bar{X} \leqslant 2.4) \approx 0.9214$ (4 d.p.)

3 a Let X be the number of heads thrown in 15 trials by one student, then $X \sim \text{B}(15, 0.25)$

$\text{E}(X) = np = 3.75$

b $\text{Var}(X) = np(p\text{-}1) = 2.8125$

By the central limit theorem $\bar{X} \approx\sim \text{N}\left(3.75, \frac{2.8125}{20}\right)$, i.e. $\bar{X} \approx\sim \text{N}(3.75, 0.1406)$

Normalising gives

$\text{P}(\bar{X} \leqslant 4) \approx 0.7475$ (4 d.p.)

4 a Let X be the number of thunderstorms hitting the town each month, then $X \sim \text{Po}(3)$

$\text{P}(X = 4) = \frac{e^{-3} 3^4}{4!} = 0.1680$ (4 d.p.)

b $\text{E}(X) = \text{Var}(X) = 3$

By the central limit theorem $\bar{X} \approx\sim \text{N}\left(3, \frac{3}{12}\right)$, i.e. $\bar{X} \approx\sim \text{N}(3, 0.25)$

$\text{P}(\bar{X} \leqslant 2.5) \approx 0.1587$ (4 d.p.)

5 $\mathrm{E}(X) = \frac{a+b}{2}$

$$= \frac{(a-3)+(3a+5)}{2}$$

$$= 2a+1$$

$$\mathrm{Var}(X) = \frac{1}{12}(b-a)^2$$

$$= \frac{1}{12}\left[(3a+5)-(a-3)\right]^2$$

$$= \frac{1}{12}(2a+8)^2$$

$$= \frac{4}{12}(a+4)^2$$

$$= \frac{(a+4)^2}{3}$$

Therefore:

$$X \sim N\left(2a+1, \frac{\frac{(a+4)^2}{3}}{40}\right)$$

$$X \sim N\left(2a+1, \frac{(a+4)^2}{120}\right)$$

6 **a** Let the discrete random variable C be the number of calls received by the telephonist in the five-minute period before her break, then $C \sim \mathrm{Po}(10)$. Let T be the total number of calls received in this period for the 30 days the telephonist records the calls, then $T = 30\bar{C}$

By the central limit theorem $\bar{C} \approx\sim \mathrm{N}\left(10, \frac{10}{30}\right)$

$$\mathrm{P}(T > 350) = \mathrm{P}\left(\bar{C} > \frac{350}{30}\right) = 1 - \mathrm{P}\left(\bar{C} < \frac{350}{30}\right) \approx 1 - 0.9981 = 0.0019 \text{ (4 d.p.)}$$

b $\mathrm{P}(C < 9) \approx 0.0416$ (4 d.p.)