Review Exercise 1

1 a $X \sim B(200, 0.02)$
$E(X) = np = 200 \times 0.02 = 4$
$Var(X) = np(1 - p) = 200 \times 0.02 \times 0.98 = 3.92$

b Because n is large and p is small.

c $X \sim B(200, 0.02)$
$Y \sim Po(4)$

$$P(Y < 6) = \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!} + \frac{e^{-4}4^5}{5!}$$
$$= e^{-4}\left(\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right)$$
$$= 0.785 \text{ (3 s.f.)}$$

2 a $X \sim B(20, 0.2)$

b $P(5 < X \leq 11) = P(X \leq 11) - P(X < 5)$
$= 0.9999 - 0.6296$
$= 0.370$ (3 s.f.)

c $P(X \geq 12) = 1 - P(X < 12)$
$= 1 - 0.998\ldots$
$= 0.0001017$
Bonus received $= 0.370 \times 1000 + 0.0001017 \times 2000$
$= \$370.20$

3 For $X \sim B(15, 0.32)$, using calculator or tables:

a $P(X = 7) = 0.101$ (3 s.f.)

b $P(X \leqslant 4) = 0.448$ (3 s.f.)

c $P(X < 8) = P(X \leqslant 7) = 0.929$ (3 s.f.)

d $P(X > 6) = 1 - P(X \leqslant 5) = 1 - 0.6607 = 0.339$ (3 s.f.)

4 a Let the random variable X denote the number of accidents per week on the stretch of motorway. The term 'rate' used in the question indicates that a Poisson distribution would be a suitable model, so $X \sim Po(1.5)$

b $P(X = 2) = \dfrac{e^{-1.5}1.5^2}{2!} = 0.2510$ (4 d.p.)
Note that as X is a discrete variable, $P(X = 2) = P(X \leqslant 2) - P(X \leqslant 1)$ and can therefore be calculated using the tables:
$P(X = 2) = P(X \leqslant 2) - P(X \leqslant 1) = 0.8088 - 0.5578 = 0.2510$

4 c $P(X \geqslant 1) = 1 - P(X = 0) = 1 - 0.2231 = 0.7769$ (from the tables)
P(at least one accident per week for 3 weeks) $= P(X \geqslant 1) \times P(X \geqslant 1) \times P(X \geqslant 1)$
$= 0.7760^3 = 0.4689$

d Let the random variable Y denote the number of accidents in a two-week period on the stretch of motorway, so $Y \sim \text{Po}(3)$
From the tables
$P(Y > 4) = 1 - P(Y \leqslant 4) = 1 - 0.8153 = 0.1847$

5 a There is no context stated, but a Poisson distribution requires an event to occur. For a Poisson distribution to be a suitable model, events should occur at a constant rate; they should occur independently or randomly; and they should occur singly.

b Let the random variable X denote the number of cars passing the point in a 60-minute period, so $X \sim \text{Po}(6)$

i $P(X = 4) = \dfrac{e^{-6} 6^4}{4!} = 0.1339$ (4 d.p.)
Note that as X is a discrete variable, $P(X = 4) = P(X \leqslant 4) - P(X \leqslant 3)$ and can therefore be calculated using the tables:
$P(X = 4) = P(X \leqslant 4) - P(X \leqslant 3) = 0.2851 - 0.1512 = 0.1339$

ii $P(X \geqslant 5) = 1 - P(X \leqslant 4)$
$= 1 - 0.2851$
$= 0.7149$
$= 0.715$ (3 s.f.)

c Let the random variable Y denote the number of cars and other vehicles passing the observation point in a 10-minute period. On average 2 other vehicles $(12 \div 6)$ pass the point so $Y \sim \text{Po}(1+2)$, i.e. $Y \sim \text{Po}(3)$
$P(Y = 1) = 3e^{-3} = 0.1494$ (4 d.p.)

Alternatively, let the random variable Z denote the number of other vehicles passing the observation point in a 10-minute period, so $Z \sim \text{Po}(2)$
P(1 car and 0 other) + P(0 car and 1 other) $= P(X = 1)P(Z = 0) + P(X = 0)P(Z = 1)$
$= e^{-1} \times e^{-2} + e^{-1} \times 2e^{-2}$
$= 0.3679 \times 0.1353 + 0.3679 \times 0.2707$
$= 0.1494$ (4 d.p.)

6 a Let the random variable X denote the number of lawn-mowers hired out by *Quikmow* in a one-hour period, so $X \sim \text{Po}(1.5)$; and let the random variable Y denote the number of lawn-mowers hired out by *Easitrim* in a one-hour period, so $Y \sim \text{Po}(2.2)$
As the variables are independent $P((X = 1) \cap (Y = 1)) = P(X = 1) \times P(Y = 1)$
$P(X = 1) \times P(Y = 1) = \dfrac{e^{-1.5} 1.5^1}{1!} \times \dfrac{e^{-2.2} 2.2^1}{1!} = 0.3347 \times 0.2438 = 0.0816$ (4 d.p.)

6 b Let the random variable Z denote the number of lawn-mowers hired out by *Quikmow* and *Easitrim* in a one-hour period, so $Z \sim \text{Po}(1.5+2.2)$, i.e. $Z \sim \text{Po}(3.7)$

$$P(Z=4)=\frac{e^{-3.7}3.7^4}{4!}=0.1931 \text{ (4 d.p.)}$$

c Let the random variable M denote the number of lawn-mowers hired out by *Quikmow* and *Easitrim* in a three-hour period, so $M \sim \text{Po}(3\times3.7)$, i.e. $M \sim \text{Po}(11.1)$

By calculator $P(M<12)=P(M\leqslant 11)=0.5673$ (4 d.p.)

7 a Mean $=\bar{x}=\frac{\sum x}{n}=\frac{290}{200}=1.45$

Variance $=\frac{\sum x^2}{n}-(\bar{x})^2=\frac{702}{200}-2.1025=1.4075$

b The fact that the mean is close to the variance supports the use of a Poisson distribution.

c Let the random variable X denote the number of toys in a cereal box and use the model, $X \sim \text{Po}(1.45)$

By calculator $P(X\geqslant 2)=1-P(X\leqslant 1)=1-0.5747=0.4253$ (4 d.p.)

8 a Let X be the number of faulty cameras.
$X \sim B(1000, 0.006)$
Using a Poisson approximation
$Y \sim \text{Po}(6)$
$P(Y>8) = 1 - P(Y\leq 8)$
$= 1 - 0.8472$
$= 0.1528$

b $P(X>t) \geq 2p$
$P(X>t) \geq 0.3056$
$P(X>7) = 0.2560$
$P(X>6) = 0.3937$
So $t = 6$

9 a Events should occur independently, in a single manner, in space or time, at a constant average rate so that the mean number in an interval is proportional to the length of the interval.

b i Let X be the number of flaws per metre.
$X \sim \text{Po}(6)$

$$P(X=0)=\frac{e^{-6}6^0}{0!}$$
$$=0.00248$$

ii Let X be the number of flaws per metre
$X \sim \text{Po}(0.6)$

$$P(X\geq 2)=1-P(X<2)$$
$$=1-\left(\frac{e^{-0.6}0.6^0}{0!}+\frac{e^{-0.6}0.6^1}{1!}\right)$$
$$=1-0.878$$
$$=0.122$$

10 a If $X \sim B(n, p)$ and n is large and p is small, then X can be approximated by Po(np).

b P(2 consecutive calls connected to wrong agent) $= 0.01 \times 0.01 = 0.0001$

c Let the random variable X denote the number of calls wrongly connected in 5 consecutive calls, so $X \sim B(5, 0.01)$

$$P(X > 1) = 1 - P(X = 1) - P(X = 0) = 1 - \binom{5}{1}(0.01)(0.99)^4 - \binom{5}{0}(0.01)^0(0.99)^5$$
$$= 1 - 0.04803 - 0.95099 = 0.00098 \text{ (5 d.p.)}$$

d Let the random variable Y denoted the number of calls wrongly connected in a day, so $Y \sim B(1000, 0.01)$

Mean $= \overline{Y} = np = 10$ Variance $= np(1-p) = 10 \times 0.99 = 9.9$

e Approximate the binomial distribution using $X \sim \text{Po}(np)$, i.e. $X \sim \text{Po}(10)$, and use tables

$\text{Po}(X > 6) = 1 - \text{Po}(X \leqslant 6) = 1 - 0.1301 = 0.8699 = 0.870$ (3 d.p.)

11 a $P(X = 3) = \binom{150}{3}(0.02)^3(0.98)^{147} = 0.2263$ (4 d.p.)

b $\lambda = np = 150 \times 0.02 = 3$

The Poisson approximation is justified in this case because n is large and p is small.

12 a $X \sim B(200, 0.015)$

b $P(X = 4) = \binom{200}{4}(0.015)^4(0.985)^{196} = 0.1693$ (4 d.p.)

c If $X \sim B(n, p)$ and n is large and p is small, then X can be approximated by Po(np), so in this case $X \approx\sim \text{Po}(200 \times 0.015)$, i.e. $X \approx\sim \text{Po}(3)$

d $P(X = 4) = \dfrac{e^{-3}3^4}{4!} = 0.1680$ (4 d.p.)

The percentage error $= \dfrac{0.1693 - 0.1680}{0.1693} \times 100 = 0.77\%$

13 $X \sim B(200, 0.01)$

Using a Poisson approximation,

$Y \sim \text{Po}(2)$

$P(1 < Y < 5) = P(Y < 5) - P(Y < 2)$

$= 0.9473 - 0.4060$

$= 0.5413$

14 Let X be the number of delayed trains.
$X \sim \text{Po}(70)$
$Y \sim \text{N}(70, 70)$
$\text{P}(Y < 75) = \text{P}(Y < 74.5)$ (apply a continuity correction)

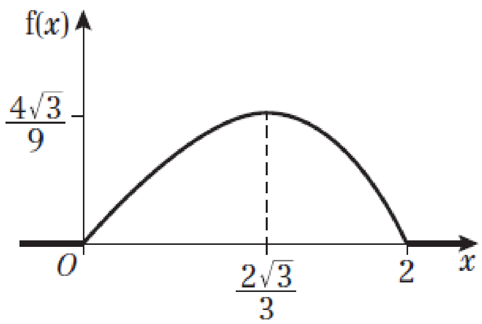
$$\text{P}(Y < 74.5) = \text{P}\left(Z < \frac{74.5 - 70}{\sqrt{70}}\right)$$
$$= \text{P}(Z < 0.5379)$$
$$= 0.7047$$

15 Let X be the number of questions answered correctly.
$X \sim \text{B}(100, 0.2)$
Using a Poisson approximation,
$Y \sim \text{Po}(20)$
$\text{P}(Y < 19) = 0.3814$

16 a The area under the probability distribution function curve must equal to 1, so:

$$\int_0^2 k(4x - x^3)\,\text{d}x = 1$$
$$\Rightarrow k\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 1$$
$$\Rightarrow k(8 - 4) = 1$$
$$\Rightarrow 4k = 1$$
$$\Rightarrow k = \frac{1}{4}$$

b Between (0, 0) and (0, 2), the function is a cubic equation with negative x^3 coefficient. In this region the function is positive, with a local maximum. (The value of x where this local maximum occurs is found in part **d**.)



c $$\text{E}(X) = \int_{-\infty}^{\infty} x\,\text{f}(x)\,\text{d}x = \int_0^2 x \times \frac{1}{4}(4x - x^3)\,\text{d}x$$
$$= \left[\frac{1}{3}x^3 - \frac{1}{20}x^5\right]_0^2 = \frac{8}{3} - \frac{32}{20} = \frac{160 - 96}{60} = \frac{64}{60} = \frac{16}{15} = 1.07 \text{ (3 s.f.)}$$

16 d The mode is the value of x at the maximum of $f(x)$, i.e. the highest point of the graph.

At the mode $f'(x) = 0$, so $1 - \frac{3}{4}x^2 = 0$

$\Rightarrow x = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.15$ (3 s.f.)

e Find the cumulative distribution function, $F(x)$:

$F(x) = \int_0^x t - \frac{1}{4}t^3 \text{d}t = \left[\frac{1}{2}t^2 - \frac{1}{16}t^4\right]_0^x = \frac{1}{2}x^2 - \frac{1}{16}x^4$ for $0 \leqslant x \leqslant 2$, and $F(x) = 0$ otherwise

Let m be the median, then $F(m) = 0.5$. This gives:

$\frac{1}{2}m^2 - \frac{1}{16}m^4 = \frac{1}{2} \Rightarrow m^4 - 8m^2 + 8 = 0$

$\Rightarrow m^2 = \frac{8 \pm \sqrt{64 - 32}}{2} = 4 \pm \sqrt{8}$, so $m^2 = 4 - \sqrt{8}$ as $0 \leqslant m^2 \leqslant 4$

$\Rightarrow m = \sqrt{1.1715\ldots} = 1.08$ (3 s.f.)

17 a The area under the probability distribution function curve must equal 1, so:

$\int_2^3 kx(x-2)\text{d}x = 1$

$\Rightarrow k\left[\frac{1}{3}x^3 - x^2\right]_2^3 = 1$

$\Rightarrow k\left(9 - 9 - \frac{8}{3} + 4\right) = 1$

$\Rightarrow \frac{4}{3}k = 1$

$\Rightarrow k = \frac{3}{4}$

b Using $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\text{d}x$ gives:

$E(X^2) = \int_2^3 \frac{3}{4}x^3(x-2)\text{d}x = \frac{3}{4}\int_2^3 (x^4 - 2x^3)\text{d}x = \frac{3}{4}\left[\frac{1}{5}x^5 - \frac{1}{2}x^4\right]_2^3$

$= \frac{3}{4}\left(\frac{243}{5} - \frac{81}{2} - \frac{32}{5} + \frac{16}{2}\right) = \frac{3}{4}\left(\frac{211}{5} - \frac{65}{2}\right) = \frac{3}{4} \times \frac{(422 - 325)}{10} = \frac{291}{40}$

So $\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{291}{40} - \left(\frac{43}{16}\right)^2 = \frac{291}{40} - \frac{1849}{256}$

$= \frac{1}{8}\left(\frac{291}{5} - \frac{1849}{32}\right) = \frac{1}{8}\left(\frac{9312 - 9245}{160}\right) = \frac{67}{8 \times 160} = \frac{67}{1280} = 0.0523$ (3 s.f.)

17 c $\mathrm{F}(x)=\int_2^x \frac{3}{4}(t^2-2t)\mathrm{d}t=\left[\frac{3}{4}\left(\frac{1}{3}t^3-t^2\right)\right]_2^x$

$$=\left(\frac{3}{4}\left(\frac{1}{3}x^3-x^2\right)-\frac{3}{4}\left(\frac{1}{3}\times 2^3-2^2\right)\right)=\frac{1}{4}(x^3-3x^2+4)$$

$$\text{So } \mathrm{F}(x)=\begin{cases} 0 & x<2 \\ \frac{1}{4}(x^3-3x^2+4) & 2\leqslant x\leqslant 3 \\ 1 & x>3 \end{cases}$$

d $\mathrm{F}(2.70)=\frac{1}{4}(2.7^3-3\times 2.7^2+4)=0.453\text{ (3 s.f.)}$

$\mathrm{F}(2.75)=\frac{1}{4}(2.75^3-3\times 2.75^2+4)=0.527\text{ (3 s.f.)}$

So $\mathrm{F}(2.70)<0.5<\mathrm{F}(2.75)$

As $\mathrm{F}(m)=0.5$, therefore the median lies between 2.70 and 2.75.

18 a Find the probability density function by differentiating: $\mathrm{F}_1'(y)=\mathrm{f}_1(y)=13-8y$

If $\mathrm{f}_1(y)$ is a probability density function, $\mathrm{f}_1(y)\geqslant 0$ on the interval $1\leqslant y\leqslant 2$

But $\mathrm{f}_1(y)<0$ when $y>\frac{13}{8}$, so $\mathrm{f}_1(y)<0$ when $1.625<y\leqslant 2$

So f_1 is not a probability density function and therefore F_1 cannot be a cumulative distribution function.

b $\mathrm{F}_2(2)=1$, so $k(2^4+2^2-2)=1$

$\Rightarrow 18k=1\Rightarrow k=\frac{1}{18}$

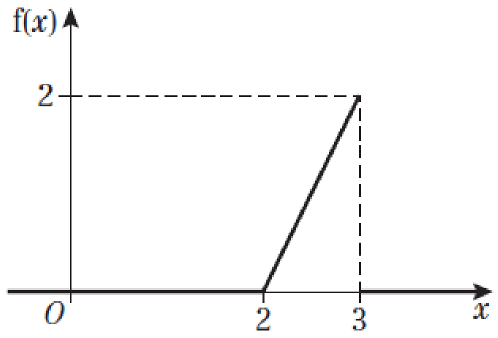
c $\mathrm{P}(Y>1.5)=1-\mathrm{P}(Y\leqslant 1.5)=1-\mathrm{F}_2(1.5)=1-\frac{1}{18}(1.5^4+1.5^2-2)$

$$=1-\frac{1}{18}\left(\frac{81}{16}+\frac{9}{4}-2\right)=1-\frac{1}{18}\left(\frac{81+36-32}{16}\right)=1-\frac{85}{288}=\frac{203}{288}=0.705\text{ (3 s.f.)}$$

d $\mathrm{f}_2(y)=\frac{\mathrm{dF}_2(y)}{\mathrm{d}y}=\frac{1}{18}\frac{\mathrm{d}}{\mathrm{d}y}(y^4+y^2-2)=\frac{1}{18}(4y^3+2y)=\frac{1}{9}(2y^3+y)$

$$\text{Hence } \mathrm{f}_2(y)=\begin{cases} \frac{1}{9}(2y^3+y) & 1\leqslant y\leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

19 a Between (2, 0) and (3, 2), f(x) is a straight line. For $x < 2$ and $x > 3$, f(x) = 0. The graph is:



b The mode is the value of x at the maximum of f(x), i.e. the highest point of the graph. So, in this case, the mode is 3.

c Using $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 \mathrm{f}(x)\mathrm{d}x$ gives:

$$\mathrm{E}(X^2) = \int_2^3 2x^2(x-2)\mathrm{d}x = \int_2^3 (2x^3 - 4x^2)\mathrm{d}x = \left[\frac{1}{2}x^4 - \frac{4}{3}x^3\right]_2^3$$

$$= \frac{81}{2} - 36 - 8 + \frac{32}{3} = \frac{32}{3} - \frac{7}{2} = \frac{64}{6} - \frac{21}{6} = \frac{43}{6}$$

So $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{43}{6} - \left(\frac{8}{3}\right)^2 = \frac{43}{6} - \frac{64}{9} = \frac{129}{18} - \frac{128}{18} = \frac{1}{18} = 0.0556$ (3 s.f.)

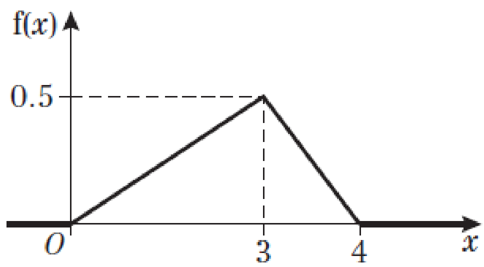
d $\mathrm{F}(m) = \int_2^m 2(x-2)\mathrm{d}x = \left[x^2 - 4x\right]_2^m = m^2 - 4m - (4-8) = m^2 - 4m + 4$

As $\mathrm{F}(m) = 0.5$, this gives $m^2 - 4m + 4 = 0.5 \Rightarrow 2m^2 - 8m + 7 = 0$

So $m = \frac{8 \pm \sqrt{64-56}}{4} = \frac{4 \pm \sqrt{2}}{4}$

As $\frac{4-\sqrt{2}}{4} < 2$, it is outside the range, so $m = \frac{4+\sqrt{2}}{4} = 2.71$ (3 s.f.)

20 a Between (0, 0) and (3, 0.5), f(x) is a straight line with a positive gradient; between (3, 0.5) and (4, 0), f(x) is a straight line with a negative gradient; and for $x < 0$ and $x > 4$, f(x) = 0.
The graph is:



b The mode is the value of x at the maximum of f(x), i.e. the highest point of the graph. So, in this case, the mode is 3.

20 c For $0 \leqslant x < 3$

$$F(x) = \int_0^x \frac{1}{6} t \, dt = \frac{1}{12} x^2$$

For $3 \leqslant x \leqslant 4$

$$F(x) = \int_3^x (2 - \frac{1}{2} t) \, dt + \int_0^3 \frac{1}{6} t \, dt = \left(2x - \frac{1}{4} x^2\right) - \left(2 \times 3 - \frac{1}{4} \times 3^2\right) + \frac{1}{12} \times 3^2 = 2x - \frac{1}{4} x^2 - 3$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{12} x^2 & 0 \leqslant x < 3 \\ 2x - \frac{1}{4} x^2 - 3 & 3 \leqslant x \leqslant 4 \\ 1 & x > 4 \end{cases}$$

d $F(3) = \frac{9}{12} = 0.75$. As $F(3) > 0.5$, the median must be between 0 and 3.

So $F(m) = \frac{1}{12} m^2 = 0.5$

$\Rightarrow m^2 = 6 \Rightarrow m = \sqrt{6} = 2.45$ (3 s.f.)

e From part **d**, P_{10} lies in $0 \leqslant x < 3$ and P_9 lies in $3 \leqslant x \leqslant 4$

$F(P_{90}) = 0.9$, so $2 P_{90} - \frac{1}{4} P_{90}^2 - 3 = \frac{9}{10}$

$\Rightarrow 5 P_{90}^2 - 40 P_{90} + 78 = 0$

$\Rightarrow P_{90} = \frac{40 \pm \sqrt{1600 - 1560}}{10} = \frac{40 \pm \sqrt{40}}{10} = 4 \pm \sqrt{0.4}$

As $4 + \sqrt{0.4} > 4$ and outside the range, $P_{90} = 4 - \sqrt{0.4} = 3.36754\ldots$

$F(P_{10}) = 0.1$, so $\frac{1}{12} P_{10}^2 = 0.1$

$\Rightarrow P_{10}^2 = 1.2 \Rightarrow P_{10} = 1.09544\ldots$

So $P_{90} - P_{10} = 3.36754 - 1.09544 = 2.27$ (3 s.f.)

21 a $P(X > 0.3) = 1 - P(X \leqslant 0.3) = 1 - F(0.3) = 1 - (2 \times 0.3^2 - 0.3^3) = 0.847$

b $F(0.59) = 2 \times (0.59)^2 - (0.59)^3 = 0.491$ (3 s.f.)

$F(0.60) = 2 \times (0.6)^2 - (0.6)^3 = 0.504$

So $F(0.59) < 0.5 < F(0.60)$

As $F(m) = 0.5$, therefore the median lies between 0.59 and 0.60.

c $f(x) = \frac{dF(x)}{dx} = \frac{d}{dx}(2x^2 - x^3) = 4x - 3x^2$ for $0 \leqslant x \leqslant 1$

$$f(x) = \begin{cases} 4x - 3x^2 & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

21 d $\mathrm{E}(X)=\int_0^1 x\,\mathrm{f}(x)\mathrm{d}x=\int_0^1(4x^2-3x^3)\mathrm{d}x=\left[4\frac{x^3}{3}-3\frac{x^4}{4}\right]_0^1=\frac{7}{12}=0.583\text{ (3 s.f.)}$

e To find the mode, solve $\mathrm{f}'(x)=0$

$\frac{\mathrm{df}(x)}{\mathrm{d}x}=4-6x$, so $\frac{\mathrm{df}(x)}{\mathrm{d}x}=0\Rightarrow x=\frac{2}{3}=0.667\text{ (3 s.f.)}$

22 a The area under the probability distribution function curve must equal 1, so:

$\int_0^2 k\mathrm{d}x+\int_2^4\frac{k}{x}\mathrm{d}x=1$

$\Rightarrow[kx]_0^2+[k\ln x]_2^4=1$

$\Rightarrow 2k+k(\ln 4-\ln 2)=1$

$\Rightarrow 2k+k\ln 2=1$

$\Rightarrow k=\frac{1}{2+\ln 2}$

b $\mathrm{E}(X)=\int_{-\infty}^{\infty}x\,\mathrm{f}(x)\mathrm{d}x=\frac{1}{2+\ln 2}\int_0^2 x\mathrm{d}x+\frac{1}{2+\ln 2}\int_2^4 1\mathrm{d}x$

$=\frac{1}{2+\ln 2}\left[\frac{1}{2}x^2\right]_0^2+\frac{1}{2+\ln 2}[x]_2^4=\frac{1}{2+\ln 2}(2+4-2)$

$=\frac{4}{2+\ln 2}=1.49\text{ (3 s.f.)}$

23 a

$$\mathrm{F}(x)=\begin{cases}0 & x<3\\ \frac{1}{49}(x^2-6x+9) & 3\le x\le 10\\ 1 & \text{otherwise}\end{cases}$$

$\mathrm{P}(X>7)=\mathrm{F}(10)-\mathrm{F}(7)$

$=1-\frac{16}{49}$

$=\frac{33}{49}$

23 b $P(X > 8) \mid P(4 < X < 9) = \dfrac{P(X > 8) \cap P(4 < X < 9)}{P(4 < X < 9)}$

$$= \frac{P(8 < X < 9)}{P(4 < X < 9)}$$

$P(4 < X < 9) = F(9) - F(4)$

$$= \frac{36}{49} - \frac{1}{49}$$

$$= \frac{35}{49}$$

$P(8 < X < 9) = 1 - F(8)$

$$= \frac{36}{49} - \frac{25}{49}$$

$$= \frac{11}{49}$$

$$\frac{P(8 < X < 9)}{P(4 < X < 9)} = \frac{\frac{11}{49}}{\frac{35}{49}} = \frac{11}{35}$$

c $f(x) = F'(x) = \dfrac{1}{49}(2x - 6)$

$$E(X) = \int_0^\infty x f(x)$$

$$= \frac{1}{49}\int_3^{10} x(2x - 6)\,dx$$

$$= \frac{1}{49}\int_3^{10} (2x^2 - 6x)\,dx$$

$$= \frac{1}{49}\left[\frac{2}{3}x^3 - 3x^2\right]_3^{10}$$

$$= \frac{1}{49}\left[\left(\frac{2}{3}(10)^3 - 3(10)^2\right) - \left(\frac{2}{3}(3)^3 - 3(3)^2\right)\right]$$

$$= \frac{1}{49}\left(\frac{1100}{3} + 9\right)$$

$$= \frac{23}{3}$$

24 a $\frac{1}{148}\int_1^5 \left(x^2+2\right)\mathrm{d}x + k\int_5^7 (3x-5)\,\mathrm{d}x = 1$

$\frac{1}{148}\left[\frac{1}{3}x^3+2x\right]_1^5 + k\left[\frac{3}{2}x^2-5x\right]_5^7 = 1$

$\frac{1}{148}\left[\left(\frac{1}{3}(5)^3+2(5)\right)-\left(\frac{1}{3}(1)^3+2(1)\right)\right]+k\left[\left(\frac{3}{2}(7)^2-5(7)\right)-\left(\frac{3}{2}(5)^2-5(5)\right)\right]=1$

$\frac{1}{148}\left[\left(\frac{155}{3}-\frac{7}{3}\right)\right]+k\left[\left(\frac{77}{2}-\frac{25}{2}\right)\right]=1$

$\frac{1}{3}+26k=1$

$26k=\frac{2}{3}$

$k=\frac{1}{39}$ as required

24 b For $x \le 1$, F(x) = 0

For $1 \le x \le 5$:

$$\mathrm{f}(x) = \frac{1}{148}\int_1^5 (x^2 + 2)\,\mathrm{d}x$$

$$\mathrm{F}(x) = \frac{1}{148}\left(\frac{1}{3}x^3 + 2x + c\right)$$

When x = 1, F(x) = 1:

$$0 = \mathrm{F}(1) = \frac{1}{148}\left(\frac{1}{3}(1)^3 + 2(1) + c\right)$$

$$= \frac{1}{148}\left(\frac{7}{3} + c\right)$$

$$c = -\frac{7}{444}$$

$$= \frac{1}{444}(x^3 + 6x - 7) \qquad 1 \le x < 5$$

For $5 \le x \le 7$:

$$\mathrm{f}(x) = \frac{1}{39}\int_5^7 (3x - 5)\,\mathrm{d}x$$

$$\mathrm{F}(x) = \frac{1}{39}\left(\frac{3}{2}x^2 - 5x + d\right)$$

When x = 7, F(x) = 1:

$$1 = \mathrm{F}(7) = \frac{1}{39}\left(\frac{3}{2}(7)^2 - 5(7) + d\right)$$

$$= \frac{1}{39}\left(\frac{77}{2} + d\right)$$

$$d = \frac{1}{2}$$

$$= \frac{1}{78}(3x^2 - 10x + 1) \qquad 5 \le x \le 7$$

F(x) = 1 $\qquad x > 7$

24 c $F(x)=\frac{1}{444}(x^3+6x-7)$ has area

$$F(x)=\frac{1}{444}\left((5)^3+6(5)-7\right)$$
$$=\frac{1}{3}$$

Therefore the median lies in the interval $5 \le x \le 7$ so

$$\frac{1}{78}(3x^2-10x+1)=\frac{1}{2}$$

$$3x^2-10x-38=0$$
$$x=\frac{10\pm\sqrt{(-10)^2-4(3)(-38)}}{2(3)}$$
$$=\frac{10\pm2\sqrt{139}}{6}$$
$$=\frac{5\pm\sqrt{139}}{3}$$

Since x must be positive

$$x=\frac{5+\sqrt{139}}{3}$$

d By similar reasoning to part **c**,

$$\frac{1}{78}(3x^2-10x+1)=\frac{4}{5}$$
$$5(3x^2-10x+1)=312$$
$$15x^2-50x-307=0$$
$$x=\frac{50\pm\sqrt{(-50)^2-4(15)(-307)}}{2(15)}$$
$$=\frac{50\pm\sqrt{20920}}{30}$$

Since x must be positive

$$x=\frac{25+\sqrt{5230}}{15}$$

Challenge

a The area under the probability distribution function curve must equal 1, so:

$$\int_0^{\infty} k\mathrm{e}^{-x}\,\mathrm{d}x = k\left[-\mathrm{e}^{-x}\right]_0^{\infty} = k(0-(-1)) = k \Rightarrow k = 1$$

b $$\int_0^{x} \mathrm{e}^{-t}\,\mathrm{d}t = \left[-\mathrm{e}^{-t}\right]_0^{x} = -\mathrm{e}^{-x} - (-1) = 1 - \mathrm{e}^{-x}$$

$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ 1-\mathrm{e}^{-x} & x \geqslant 0 \end{cases}$$

c $$\mathrm{P}(1 < X < 4) = \mathrm{P}(X < 4) - \mathrm{P}(X < 1) = \mathrm{F}(4) - \mathrm{F}(1)$$

$$= (1-\mathrm{e}^{-4}) - (1-\mathrm{e}^{-1}) = \mathrm{e}^{-1} - \mathrm{e}^{-4} = \frac{\mathrm{e}^{3}-1}{\mathrm{e}^{4}}$$