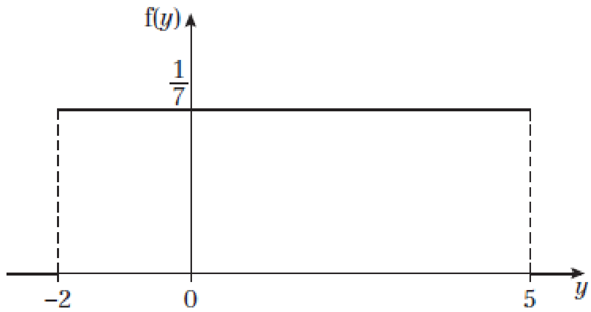
Chapter Review 5

1 a $b-a=5-(-2)=7$, so the graph is a straight line from $\left(-2,\frac{1}{7}\right)$ to $\left(5,\frac{1}{7}\right)$ and 0 otherwise.

The sketch of the graph is:



b $\text{E}(X)=\frac{a+b}{2}=\frac{-2+5}{2}=1.5$

c $\text{Var}(X)=\frac{(b-a)^2}{12}=\frac{(5-(-2))^2}{12}=\frac{49}{12}$

d For $-2\leqslant x\leqslant 5$, $\text{F}(x)=\int_{-2}^{x}\frac{1}{b-a}\,\text{d}t=\int_{-2}^{x}\frac{1}{7}\,\text{d}t=\left[\frac{t}{7}\right]_{-2}^{x}=\frac{x}{7}+\frac{2}{7}=\frac{x+2}{7}$

So:

$$\text{F}(x)=\begin{cases}0 & x<-2\\ \frac{x+2}{7} & -2\leqslant x\leqslant 5\\ 1 & x>5\end{cases}$$

e $\text{P}(3.5<X<5.5)=(5-3.5)\times\frac{1}{7}=\frac{3}{14}$

Note that as $\text{f}(x)=0$ for $x>5$, $\text{P}(5<X<5.5)=0$

f As X is a continuous random variable $\text{P}(X=n)=0$ for any discrete number n.
So $\text{P}(X=4)=0$

g $\text{P}(X>0\,|\,X<2)=\frac{\text{P}(X>0\cap X<2)}{\text{P}(X<2)}=\frac{\text{P}(0<X<2)}{\text{P}(X<2)}=\frac{\frac{2}{7}}{\frac{4}{7}}=\frac{1}{2}$

h $\text{P}(X>3\,|\,X>0)=\frac{\text{P}(X>3\cap X>0)}{\text{P}(X>0)}=\frac{\text{P}(X>3)}{\text{P}(X>0)}=\frac{\frac{2}{7}}{\frac{5}{7}}=\frac{2}{5}$

2 a The area under the probability density function graph must be 1, so:
$0.2(k-(-4))=1\Rightarrow k=5-4=1$

b $\text{P}(-2<X<-1)=(-1-(-2))\times\frac{1}{1-(-4)}=\frac{1}{5}=0.2$

2 c $\mathrm{E}(X)=\dfrac{a+b}{2}=\dfrac{-4+1}{2}=-\dfrac{3}{2}=-1.5$

d $\mathrm{Var}(X)=\dfrac{(b-a)^2}{12}=\dfrac{(1-(-4))^2}{12}=\dfrac{25}{12}$

e For $-4\leqslant x\leqslant 1$, $\mathrm{F}(x)=\int_{-4}^{x}\dfrac{1}{b-a}\,\mathrm{d}t=\int_{-4}^{x}\dfrac{1}{5}\,\mathrm{d}t=\left[\dfrac{t}{5}\right]_{-4}^{x}=\dfrac{x}{5}+\dfrac{4}{5}=\dfrac{x+4}{5}$

So:

$$\mathrm{F}(x)=\begin{cases}0 & x<-4\\ \dfrac{x+4}{5} & -4\leqslant x\leqslant 1\\ 1 & x>1\end{cases}$$

3 a $\mathrm{E}(Y)=\dfrac{a+b}{2}=2\Rightarrow a+b=4\Rightarrow a=4-b$

$\mathrm{Var}(Y)=\dfrac{(b-a)^2}{12}=3\Rightarrow(b-a)^2=36$

So $(b-4+b)^2=36$ substituting for a

$\Rightarrow(2b-4)^2=36$

$\Rightarrow 2b-4=\pm 6$

$\Rightarrow b=2\pm 3$

$b=-1\Rightarrow a=5$ reject as $b>a$

So solution is $b=5\Rightarrow a=-1$

b $\mathrm{P}(Y>1.8)=\dfrac{5-1.8}{5-(-1)}=\dfrac{3.2}{6}=\dfrac{8}{15}=0.533$ (3 s.f.)

4 a As $10-5\times 0=10$ and $10-5\times 2=0$

So $Y\sim\mathrm{U}[0,10]$

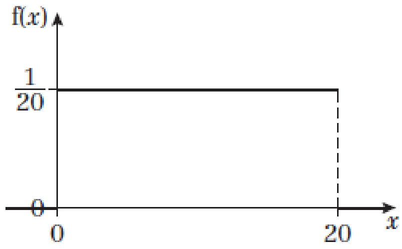
b $\mathrm{P}(Y<3)=\dfrac{3-0}{10-0}=\dfrac{3}{10}=0.3$

c $\mathrm{P}(Y>3\mid X>0.5)=\mathrm{P}(Y>3\mid Y<7.5)=\dfrac{\mathrm{P}(Y>3\cap Y<7.5)}{\mathrm{P}(Y<7.5)}=\dfrac{\mathrm{P}(3<Y<7.5)}{\mathrm{P}(Y<7.5)}$

$=\dfrac{7.5-3}{7.5-0}=\dfrac{4.5}{7.5}=\dfrac{9}{15}=\dfrac{3}{5}=0.6$

5 **a** The variable X has a continuous uniform distribution, $X \sim \mathrm{U}[0,20]$.

The graph is a straight line from $\left(0,\frac{1}{20}\right)$ to $\left(20,\frac{1}{20}\right)$ and otherwise 0.



b $\mathrm{E}(X)=\frac{a+b}{2}=\frac{0+20}{2}=10$

$\mathrm{Var}(X)=\frac{(b-a)^2}{12}=\frac{20^2}{12}=\frac{400}{12}=\frac{100}{3}$

c The string with the mark could be the shorter piece ($X < 10$) or the longer piece ($X > 10$)
Require the shorter length of string to be > 8
If the string with the mark on it is the shorter length of string, then require $8 < X < 10$
If the string with the mark on it is the longer length of string, then require $10 < X < 12$
So the required probability is:

$$\mathrm{P}(8 < X < 10) \cup \mathrm{P}(10 < X < 12) = \mathrm{P}(8 < X < 12) = \frac{4}{20} = 0.2$$

6 **a** The temperature is between 28.5 °C and 29.5 °C, so X can be any value between –0.5 °C and 0.5 °C with equal probability. So a suitable model is $X \sim \mathrm{U}[-0.5, 0.5]$

b $\mathrm{P}(-0.2 < X < 0.2)=\frac{0.2-(-0.2)}{0.5-(-0.5)}=0.4$

c $\mathrm{Var}(X)=\frac{(b-a)^2}{12}=\frac{(0.5-(-0.5))^2}{12}=\frac{1}{12}$

7 **a** $X \sim \mathrm{U}[-3, 10]$
The probability density function of X is:

$$\mathrm{f}(x)=\begin{cases}\frac{1}{13} & -3 \leqslant x \leqslant 10 \\ 0 & \text{otherwise}\end{cases}$$

b $\text{Mean} = \mathrm{E}(X)=\frac{a+b}{2}=\frac{-3+10}{2}=3.5$ minutes

7 c For $-3 \leqslant x \leqslant 10$, $\mathrm{F}(x)=\int_{-3}^{x}\frac{1}{b-a}\,\mathrm{d}t=\int_{-3}^{x}\frac{1}{13}\,\mathrm{d}t=\left[\frac{t}{13}\right]_{-3}^{x}=\frac{x}{13}+\frac{3}{13}=\frac{x+3}{13}$

So:

$$\mathrm{F}(x)=\begin{cases}0 & x<-3\\ \frac{x+3}{13} & -3 \leqslant x \leqslant 10\\ 1 & x>10\end{cases}$$

d $\mathrm{P}(5<X<10)=(10-5)\times\frac{1}{13}=\frac{5}{13}$

8 a The difference between the true length and the measured length could be any value between –0.5 cm and 0.5 cm with equal probability. So a suitable model is $X \sim \mathrm{U}[-0.5, 0.5]$

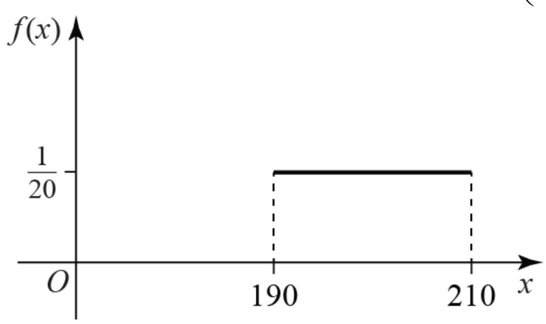
b $\mathrm{P}(-0.2<X<0.2)=\frac{0.2-(-0.2)}{0.5-(-0.5)}=0.4$

c P(3 pipes between –0.2 and 0.2) = 0.4^3 = 0.064

9 a The volume can be any value between 190 ml and 210 ml with equal probability. So a suitable model is $X \sim \mathrm{U}[190, 210]$. The probability density function is:

$$\mathrm{f}(x)=\begin{cases}\frac{1}{20} & 190 \leqslant x \leqslant 210\\ 0 & \text{otherwise}\end{cases}$$

The graph of f(x) is a straight line from $\left(190, \frac{1}{20}\right)$ to $\left(210, \frac{1}{20}\right)$ and otherwise 0.



b i $\mathrm{P}(X<198)=\frac{198-190}{20}=\frac{8}{20}=\frac{2}{5}=0.4$

ii As X is a continuous random variable P($X = n$) = 0 for any discrete number n.
So P(X = 198) =0

9 c The cumulative distribution function is:

$$F(x)=\begin{cases}0 & x<190\\ \dfrac{x-190}{20} & 190\leqslant x\leqslant 210\\ 1 & x>210\end{cases}$$

$F(Q_3)=0.75 \Rightarrow Q_3=205 \qquad F(Q_1)=0.25 \Rightarrow Q_1=195$

So $Q_3-Q_1=205-195=10$

d $P(X>200\mid X>195)=\dfrac{P(X>195\cap X>200)}{P(X>195)}=\dfrac{P(X>200)}{P(X>195)}$

$=\dfrac{0.5}{0.75}=\dfrac{2}{3}$

10 a Continuous uniform distribution. The difference between the true length and the measured length could be any value between –0.5 cm and 0.5 cm with equal probability. So a suitable model is $X \sim U[-0.5, 0.5]$

b Normal distribution.

11 a As $5b - b = 4b$, the probability density function is

$$f(x)=\begin{cases}\dfrac{1}{4b} & b\leqslant x\leqslant 5b\\ 0 & \text{otherwise}\end{cases}$$

b $E(X)=\dfrac{b+5b}{2}=3b$

c $\operatorname{Var}(X)=\int x^2 f(x)dx-(E(X))^2=\int_b^{5b}\dfrac{x^2}{4b}dx-(3b)^2$

$=\left[\dfrac{x^3}{12b}\right]_b^{5b}-9b^2=\dfrac{125b^2-b^2}{12}-\dfrac{108b^2}{12}=\dfrac{16b^2}{12}=\dfrac{4b^2}{3}$

d $P(X>10)=\dfrac{15-10}{12}=\dfrac{5}{12}$

e Let the random variable Y be the number of times that $X > 10$ in five observations, so Y has a binomial distribution, $Y \sim B\left(5, \dfrac{5}{12}\right)$

So $P(Y=3)=\binom{5}{3}\left(\dfrac{5}{12}\right)^3\left(\dfrac{7}{12}\right)^2=0.246$ (3 s.f.)

Challenge

a As θ can take any value between 0 and 2π with equal probability, a continuous uniform distribution would be suitable, with $\theta \sim \text{U}[0, 2\pi]$

b $X = r|\sin\theta|$

So $\text{E}(X) = \text{E}\left(r|\sin\theta|\right) = \int_0^{2\pi} r|\sin\theta|\frac{1}{2\pi}\,\text{d}\theta$

$= \frac{r}{2\pi}\left(\int_0^{\pi}\sin\theta\,\text{d}\theta + \int_{\pi}^{2\pi} -\sin\theta\,\text{d}\theta\right)$

$= \frac{r}{2\pi}\left(\left[-\cos\theta\right]_0^{\pi} + \left[\cos\theta\right]_{\pi}^{2\pi}\right)$

$= \frac{r}{2\pi}(2+2)$

$= \frac{2r}{\pi} \approx 0.6366r$

c Spin the spinner 100 times and measure X each time. Take the mean of these observations and divide $2r$ by this value.