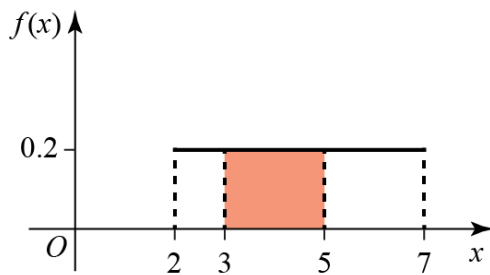**Statistics 2**   Solution Bank   P Pearson

Exercise 5A

1 a $\dfrac{1}{b-a} = \dfrac{1}{7-2} = 0.2$

So a sketch of the probability distribution function (with the required probability shaded) is:

$P(3 < X < 5) = (5-3) \times 0.2 = 0.4$

b $P(X > 4) = (7-4) \times 0.2 = 0.6$

2 a Area under the probability distribution function curve is 1, so:

$$(k - 2.6) \times 0.1 = 1$$
$$(k - 2.6) = 10$$
$$k = 12.6$$

b $P(4 < X < 7.9) = (7.9 - 4) \times 0.1 = 0.39$

3 a Area under the probability distribution function curve is 1, so:

$$k \times (6 - (-2)) = 1$$
$$8k = 1$$
$$k = \dfrac{1}{8}$$

b $P(-1.3 < X < 4.2) = \dfrac{1}{8} \times (4.2 - (-1.3)) = \dfrac{1}{8} \times 5.5 = 0.6875$

c $3X < X + p \Rightarrow X < \dfrac{p}{2}$

$P\left(X < \dfrac{p}{2}\right) = 0.5$
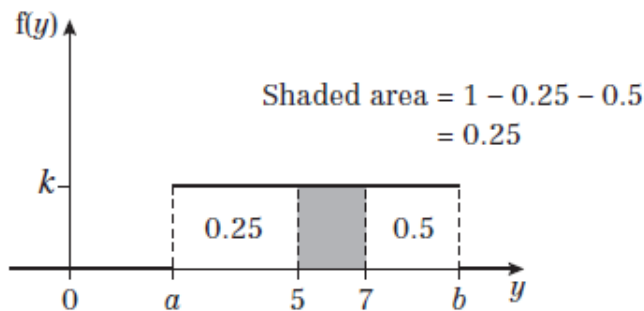
$\Rightarrow \dfrac{p}{2} = 2 \Rightarrow p = 4$

d Using $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$P(X > 5 \mid X > 0) = \dfrac{P(X > 5 \cap X > 0)}{P(X > 0)} = \dfrac{P(X > 5)}{P(X > 0)} = \dfrac{\frac{1}{8}}{\frac{6}{8}} = \dfrac{1}{6}$

3 e $P(X > 0 \mid X < 3) = \dfrac{P(X > 0 \cap X < 3)}{P(X < 3)} = \dfrac{P(0 < X < 3)}{P(X < 3)} = \dfrac{\frac{3}{8}}{\frac{5}{8}} = \dfrac{3}{5}$

f $P(X < 1 \mid 0 < X < 2) = \dfrac{P(X < 1 \cap 0 < X < 2)}{P(0 < X < 2)} = \dfrac{P(0 < X < 1)}{P(0 < X < 2)} = \dfrac{\frac{1}{8}}{\frac{2}{8}} = \dfrac{1}{2}$

4 Sketching the probability distribution function:

$P(5 < Y < 7) = 1 - P(Y < 5) - P(Y > 7) = 1 - 0.25 - 0.5 = 0.25$

But from the sketch, $P(5 < Y < 7) = (7 - 5) \times k = 2k$

So $2k = 0.25 \Rightarrow k = 0.125 = \dfrac{1}{8}$

$P(X > 7) = (b - 7)k = \dfrac{(b - 7)}{8} = \dfrac{1}{2} \Rightarrow b = 4 + 7 = 11$

$P(X < 5) = (5 - a)k = \dfrac{(5 - a)}{8} = \dfrac{1}{4} \Rightarrow a = 5 - 2 = 3$

5 a As $2 \times 2 + 5 = 9$ and $2 \times 8 + 5 = 21$
Hence $Y \sim U[9, 21]$

b For Y, $\dfrac{1}{b - a} = \dfrac{1}{21 - 9} = \dfrac{1}{12}$

$P(12 < Y < 20) = (20 - 12) \times \dfrac{1}{12} = \dfrac{2}{3}$

6 a Continuous uniform distribution

b As $20 - 2 \times 2 = 16$ and $20 - 2 \times 12 = -4$
Hence $Y \sim U[-4, 16]$

By symmetry, $E(Y) = \dfrac{16 + (-4)}{2} = \dfrac{12}{2} = 6$

c For Y, $\dfrac{1}{b - a} = \dfrac{1}{16 - (-4)} = \dfrac{1}{20}$

$P(Y < 4) = (4 - (-4)) \times \dfrac{1}{20} = \dfrac{8}{20} = \dfrac{2}{5}$

6 d If $X < 10$, then $Y > 0$, so

$$P(Y > 4 \mid X < 10) = P(Y > 4 \mid Y > 0) = \frac{P(Y > 4 \cap Y > 0)}{P(Y > 0)} = \frac{P(Y > 4)}{P(Y > 0)} = \frac{\frac{12}{20}}{\frac{16}{20}} = \frac{12}{16} = \frac{3}{4}$$

7 a $E(X) = \dfrac{5 + (-3)}{2} = 1$

b $Var(X) = \dfrac{(b-a)^2}{12} = \dfrac{(5-(-3))^2}{12} = \dfrac{64}{12} = \dfrac{16}{3}$

c $Var(X) = E(X^2) - (E(X))^2$, so $E(X^2) = \dfrac{16}{3} + 1^2 = \dfrac{19}{3}$

d For $-3 \leqslant x \leqslant 5$, $F(x) = \displaystyle\int_{-3}^{x} \frac{1}{b-a} \, dt = \int_{-3}^{x} \frac{1}{8} \, dt = \left[\frac{t}{8}\right]_{-3}^{x} = \frac{x}{8} + \frac{3}{8} = \frac{x+3}{8}$

So the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < -3 \\ \dfrac{x+3}{8} & -3 \leqslant x \leqslant 5 \\ 1 & x > 5 \end{cases}$$

8 a $E(X) = \dfrac{a+b}{2} = \dfrac{5+1}{2} = 3$ $Var(X) = \dfrac{(b-a)^2}{12} = \dfrac{(5-1)^2}{12} = \dfrac{16}{12} = \dfrac{4}{3}$

b $E(X) = \dfrac{a+b}{2} = \dfrac{6+(-2)}{2} = 2$ $Var(X) = \dfrac{(b-a)^2}{12} = \dfrac{(6-(-2))^2}{12} = \dfrac{64}{12} = \dfrac{16}{3}$

9 a $E(X) = \dfrac{5.5 + 3.5}{2} = 4.5$

b $Var(X) = \dfrac{(5.5 - 3.5)^2}{12} = \dfrac{4}{12} = \dfrac{1}{3}$

c $Var(X) = E(X^2) - (E(X))^2$

So $\dfrac{1}{3} = E(X^2) - 4.5^2$

$\Rightarrow E(X^2) = 20.25 + \dfrac{1}{3} = \dfrac{247}{12} = 20.6$ (3 s.f.)

9 d For $3.5 \leqslant x \leqslant 5.5$, $F(x) = \int_{3.5}^{x} \dfrac{1}{b-a}\,dt = \int_{3.5}^{x} \dfrac{1}{2}\,dt = \left[\dfrac{t}{2}\right]_{3.5}^{x} = \dfrac{x}{2} - \dfrac{7}{4} = \dfrac{2x-7}{4}$

So the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 3.5 \\ \dfrac{2x-7}{4} & 3.5 \leqslant x \leqslant 5.5 \\ 1 & x > 5.5 \end{cases} = \begin{cases} 0 & x < 3.5 \\ 0.5x - 1.75 & 3.5 \leqslant x \leqslant 5.5 \\ 1 & x > 5.5 \end{cases}$$

10 $E(Y) = \dfrac{a+b}{2} = 1 \Rightarrow b = 2 - a$ **(1)**

$Var(Y) = \dfrac{(b-a)^2}{12} = \dfrac{4}{3} \Rightarrow (b-a)^2 = 16$ **(2)**

Substituting equation **(1)** in equation **(2)** gives:

$(2 - a - a)^2 = 16$

$\Rightarrow 2 - 2a = \pm 4$

$\Rightarrow a = 3$ or -1

If $a = 3$, $b = 2 - 3 = -1$

If $a = -1$, $b = 2 - (-1) = 3$

As $b > a$, the solution is $a = -1$ and $b = 3$

11 $E(X) = \dfrac{5 + (-1)}{2} = 2$

$Var(X) = \dfrac{(5 - (-1))^2}{12} = 3$

$E(Y) = E(4X - 6) = 4E(X) - 6 = 8 - 6 = 2$

$Var(Y) = Var(4X - 6) = 4^2 Var(X) = 16 \times 3 = 48$

12 a $E(R) = \dfrac{\alpha + \beta}{2} = 5 \Rightarrow \beta = 10 - \alpha$

$Var(R) = \dfrac{(\beta - \alpha)^2}{12} = \dfrac{4}{3} \Rightarrow (10 - 2\alpha)^2 = 16 \Rightarrow \alpha = 5 \pm 2 \Rightarrow \alpha = 3$ or 7

If $\alpha = 7$, $\beta = 3$;   if $\alpha = 3$, $\beta = 7$

But $\alpha < \beta$ so the solution is $\alpha = 3$, $\beta = 7$

b $P(R < 5.2) = \dfrac{5.2 - \alpha}{\beta - \alpha} = \dfrac{2.2}{4} = 0.55$

13 a The probability density function of X is

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \alpha \leqslant x \leqslant \beta \\ 0 & \text{otherwise} \end{cases}$$

13 b $E(X) = \dfrac{\alpha + \beta}{2} = 2.5 \Rightarrow \beta = 5 - \alpha$

$P(X < 1) = \dfrac{1 - \alpha}{\beta - \alpha} = \dfrac{4}{11} \Rightarrow \dfrac{1 - \alpha}{5 - 2\alpha} = \dfrac{4}{11}$ substituting for β
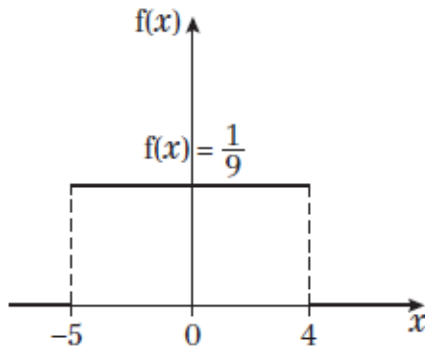
$\Rightarrow 11 - 11\alpha = 20 - 8\alpha$

$\Rightarrow 3\alpha = -9$

$\Rightarrow \alpha = -3, \beta = 5 - \alpha = 8$

14 a   The probability density function of X is

$f(x) = \begin{cases} \dfrac{1}{9} & -5 \leqslant x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$

b   Sketching the probability distribution function gives:

c $E(X) = \dfrac{a + b}{2} = \dfrac{4 + (-5)}{2} = -\dfrac{1}{2}$ $\text{Var}(X) = \dfrac{(b - a)^2}{12} = \dfrac{(4 - (-5))^2}{12} = \dfrac{81}{12} = \dfrac{27}{4}$

$\text{Var}(X) = E(X^2) - (E(X))^2$

So $\dfrac{27}{4} = E(X^2) - \dfrac{1}{2^2} \Rightarrow E(X^2) = \dfrac{28}{4} = 7$

d $P(-0.2 < X < 0.6) = \dfrac{0.6 - (-0.2)}{4 - (-5)} = \dfrac{0.8}{9} = \dfrac{4}{45}$

15 a $P(X < 0) = F(0) = \dfrac{3}{7}$

b $f(x) = \dfrac{d}{dx} F(x)$

So the probability density function is:

$f(x) = \begin{cases} \dfrac{1}{7} & -3 \leqslant x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$

c $X \sim U[-3, 4]$, it has a continuous uniform distribution

15 d $E(X) = \dfrac{a+b}{2} = \dfrac{4+(-3)}{2} = \dfrac{1}{2}$

$Var(X) = \dfrac{(b-a)^2}{12} = \dfrac{(4-(-3))^2}{12} = \dfrac{49}{12}$

16 a $E(X) = \dfrac{4+(-1)}{2} = \dfrac{3}{2} = 1.5$

b $Var(X) = \dfrac{(4-(-1))^2}{12} = \dfrac{25}{12}$

c $Var(X) = E(X^2) - (E(X))^2$

So $\dfrac{25}{12} = E(X^2) - \dfrac{9}{4} \Rightarrow E(X^2) = \dfrac{25+27}{12} = \dfrac{52}{12} = \dfrac{13}{3}$

d $P(X < 1.4) = \dfrac{1.4-(-1)}{4-(-1)} = \dfrac{2.4}{5} = 0.48$

e Let the random variable Y represent the number of times that X is observed to be less than 1.4 in 6 observations, then using the result from part **d**, $Y \sim B(6, 0.48)$

$P(Y = 4) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}(0.48)^4(1-0.48)^2 = \dfrac{6 \times 5}{2} \times 0.48^4 \times 0.52^2 = 0.2153 \text{ (4 d.p.)}$

17 a $E(X) = \dfrac{\alpha+\beta}{2} = 7.5 \Rightarrow \beta = 15 - \alpha$

$P(X \leqslant 10.5) = 1 - P(X > 10.5) = 1 - 0.25 = 0.75$

$P(X \leqslant 10.5) = \dfrac{10.5-\alpha}{\beta-\alpha} = 0.75 \Rightarrow \dfrac{10.5-\alpha}{15-2\alpha} = 0.75$ substituting for β

$\Rightarrow 10.5 - \alpha = 11.25 - 1.5\alpha$

$\Rightarrow \alpha = 1.5, \ \beta = 15 - \alpha = 13.5$

b i $P(x < c) = \dfrac{c-\alpha}{\beta-\alpha} = \dfrac{c-1.5}{12} = \dfrac{1}{3} \Rightarrow c = 5.5$

ii $P(c < X < 9) = P(5.5 < X < 9) = \dfrac{3.5}{12} = \dfrac{7}{24}$

Challenge

a $P\left(X < \dfrac{3}{7}\alpha + \dfrac{4}{7}\beta\right) = \dfrac{\dfrac{3}{7}\alpha + \dfrac{4}{7}\beta - \alpha}{\beta-\alpha} = \dfrac{\dfrac{4}{7}(\beta-\alpha)}{\beta-\alpha} = \dfrac{4}{7}$

b $P\left(X > \dfrac{2}{5}\alpha + \dfrac{3}{5}\beta\right) = \dfrac{\beta - \dfrac{2}{5}\alpha - \dfrac{3}{5}\beta}{\beta-\alpha} = \dfrac{\dfrac{2}{5}(\beta-\alpha)}{\beta-\alpha} = \dfrac{2}{5}$