Chapter Review 4

1 a $E(X) = \int x f(x)dx = \int_0^2 \frac{x}{3}\left(1+\frac{x}{2}\right) dx = \int_0^2 \frac{x}{3}+\frac{x^2}{6} dx$

$= \left[\frac{x^2}{6}+\frac{x^3}{18}\right]_0^2 = \frac{2^2}{6}+\frac{2^3}{18} = \frac{2}{3}+\frac{4}{9} = \frac{10}{9}$

$E(3X+2) = 3E(X)+2 = 3\times\frac{10}{9}+2 = \frac{30+18}{9} = \frac{48}{9} = \frac{16}{3}$

b $\mathrm{Var}(X) = \int x^2 f(x)dx - (E(X))^2 = \int_0^2 \frac{x^2}{3}\left(1+\frac{x}{2}\right)dx - \left(\frac{10}{9}\right)^2 = \int_0^2 \frac{x^2}{3}+\frac{x^3}{6} dx - \frac{100}{81}$

$= \left[\frac{x^3}{9}+\frac{x^4}{24}\right]_0^2 - \frac{100}{81} = \frac{2^3}{9}+\frac{2^4}{24}-\frac{100}{81} = \frac{8}{9}+\frac{2}{3}-\frac{100}{81} = \frac{72+54-100}{81} = \frac{26}{81}$

$\mathrm{Var}(3X+2) = 3^2\mathrm{Var}(X) = 9\times\frac{26}{81} = \frac{26}{9}$

c $P(X<1) = \int_0^1 \frac{1}{3}\left(1+\frac{x}{2}\right) dx = \int_0^1 \frac{1}{3}+\frac{x}{6}dx$

$= \left[\frac{x}{3}+\frac{x^2}{12}\right]_0^1 = \frac{1}{3}+\frac{1}{12} = \frac{5}{12}$

d $P(X > E(X)) = P\left(X > \frac{10}{9}\right) = 1 - P\left(X < \frac{10}{9}\right)$

$= 1 - \int_0^{\frac{10}{9}} \frac{1}{3}\left(1+\frac{x}{2}\right)dx = 1 - \int_0^{\frac{10}{9}} \frac{1}{3}+\frac{x}{6}dx = 1 - \left[\frac{x}{3}+\frac{x^2}{12}\right]_0^{\frac{10}{9}}$

$= 1 - \left(\frac{10}{27}+\frac{100}{972}\right) = 1 - \left(\frac{90}{243}+\frac{25}{243}\right) = 1 - \frac{115}{243} = \frac{128}{243}$

e $P(0.5 < X < 1.5) = \int_{0.5}^{1.5}\left(\frac{1}{3}+\frac{x}{6}\right) dx = \left[\frac{x}{3}+\frac{x^2}{12}\right]_{0.5}^{1.5}$

$= \left(\frac{1.5}{3}+\frac{1.5^2}{12}\right) - \left(\frac{0.5}{3}+\frac{0.5^2}{12}\right) = \frac{3}{6}+\frac{9}{48}-\frac{1}{6}-\frac{1}{48}$

$= \frac{2}{6}+\frac{8}{48} = \frac{2}{6}+\frac{1}{6} = \frac{3}{6} = 0.5$

2 a $E(X) = \int x f(x)dx = \int_0^1 2x-2x^2 \, dx = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = 1-\frac{2}{3} = \frac{1}{3}$

2 b $\mathrm{Var}(X) = \int x^2 \,\mathrm{f}(x)\mathrm{d}x - (\mathrm{E}(X))^2 = \int_0^1 2x^2 - 2x^3 \mathrm{d}x - \left(\frac{1}{3}\right)^2$

$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1 - \frac{1}{9} = \frac{2}{3} - \frac{1}{2} - \frac{1}{9} = \frac{12-9-2}{18} = \frac{1}{18}$$

c $\mathrm{E}(2X+1) = 2\mathrm{E}(X) + 1 = 2 \times \frac{1}{3} + 1 = \frac{5}{3}$

$\mathrm{Var}(2X+1) = 2^2 \mathrm{Var}(X) = \frac{4}{18} = \frac{2}{9}$

d ***Method 1***

$\mathrm{F}(x) = \int_0^x (2-2t)\ \mathrm{d}t = \left[2t - t^2\right]_0^x = 2x - x^2$

Method 2

$\mathrm{F}(x) = \int 2 - 2x\mathrm{d}x = 2x - x^2 + c$

$\mathrm{F}(2) = 1,\ \text{so } 2 - 1 + c = 1 \Rightarrow c = 0$

So the cumulative distribution function is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ 2x - x^2 & 0 \leqslant x \leqslant 1 \\ 1 & x > 1 \end{cases}$$

e $\mathrm{F}(m) = 0.5,\ \text{so } 2m - m^2 = 0.5 \Rightarrow 2m^2 - 4m + 1 = 0$

$m = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{\sqrt{2}}{2}$

As $0 \leqslant m \leqslant 1,\ m = 1 - \frac{\sqrt{2}}{2} = 0.293$ (3 s.f.)

3 a As $\mathrm{F}(2) = 1,\ \mathrm{F}(2) = k(4-2) = 1 \Rightarrow k = \frac{1}{2}$

b $\mathrm{P}(Y < 1.5) = \mathrm{F}(1.5) = \frac{1}{2} \times (1.5^2 - 1.5) = \frac{1}{2}\left(\frac{9}{4} - \frac{3}{2}\right) = \frac{3}{8} = 0.375$

c $\mathrm{F}(m) = 0.5,\ \text{so } \frac{1}{2}(m^2 - m) = 0.5 \Rightarrow m^2 - m - 1 = 0$

$m = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

As $1 \leqslant m \leqslant 2,\ m = \frac{1+\sqrt{5}}{2} = 1.62$ (3 s.f.)

3 d Using $\dfrac{d}{dy}F(y)=f(y)$

$$\frac{d}{dy}\left(\frac{1}{2}(y^2-y)\right)=y-\frac{1}{2}$$

So the probability density function is:

$$f(y)=\begin{cases} y-\dfrac{1}{2} & 1\leqslant y\leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

4 a $P(X>2.4)=1-P(X<2.4)=1-F(2.4)=1-\dfrac{1}{5}(2.4^2-4)=0.648$

Alternative method:

$P(X>2.4)=P(X<3)-P(X<2.4)=F(3)-F(2.4)$

$$=\frac{1}{5}(3^2-4)-\frac{1}{5}(2.4^2-4)=0.648$$

b $F(m)=0.5$, so $\dfrac{1}{5}(m^2-4)=0.5$

$\Rightarrow 2(m^2-4)=5$ multiplying both sides by 10

$\Rightarrow 2m^2=13$

$\Rightarrow m=\sqrt{\dfrac{13}{2}}=2.55$ (3 s.f.) as $-\sqrt{\dfrac{13}{2}}$ is not in the range $2\leqslant m\leqslant 3$

c Using $\dfrac{d}{dx}F(x)=f(x)$

$$\frac{d}{dx}\left(\frac{1}{5}(x^2-4)\right)=\frac{2x}{5}$$

So the probability density function is:

$$f(x)=\begin{cases} \dfrac{2x}{5} & 2\leqslant x\leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

d $E(X)=\displaystyle\int x\,f(x)dx=\int_2^3\frac{2}{5}x^2\ dx=\left[\frac{2}{15}x^3\right]_2^3=\frac{2}{15}(27-8)=\frac{38}{15}$

e The mode occurs at the maximum point of the probability density function graph. As f(x) is strictly increasing on the interval [2, 3] and 0 elsewhere, the mode must be 3.

5 a The area under the probability density function graph must be 1, so:

$$\int_0^2 kx^2\,\mathrm{d}x = 1 \Rightarrow \left[\frac{kx^3}{3}\right]_0^2 = 1$$

So $\frac{8k}{3} = 1 \Rightarrow k = \frac{3}{8}$

b $\mathrm{E}(X) = \int x\,\mathrm{f}(x)\mathrm{d}x = \int_0^2 \frac{3x^3}{8}\mathrm{d}x = \left[\frac{3x^4}{32}\right]_0^2 = \frac{3}{2} = 1.5$

c ***Method 1***

$$\mathrm{F}(x) = \int_0^x \frac{3}{8}t^2\,\mathrm{d}t = \left[\frac{1}{8}t^3\right]_0^x = \frac{x^3}{8}$$

Method 2

$$\mathrm{F}(x) = \int \frac{3}{8}x^2\mathrm{d}x = \frac{x^3}{8} + c$$

$\mathrm{F}(2) = 1$, so $\frac{2^3}{8} + c = 1 \Rightarrow c = 0$

So the cumulative distribution function is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ \frac{x^3}{8} & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

d $\mathrm{F}(m) = 0.5$, so $\frac{m^3}{8} = 0.5$

$\Rightarrow m^3 = 4 \;\Rightarrow m = 1.59$ (3 s.f.)

e The mode occurs at the maximum point of the probability density function graph. As f(x) is strictly increasing on the interval [0, 2] and 0 elsewhere, the mode must be 2.

6 a The area under the probability density function graph must be 1, so:

$$\int_1^3 k(y^2 + 2y + 2)\,\mathrm{d}y = 1$$

$$\Rightarrow \left[k\left(\frac{y^3}{3} + y^2 + 2y\right)\right]_1^3 = 1$$

$$\Rightarrow k\left(\frac{3^3}{3} + 3^2 + 6\right) - k\left(\frac{1}{3} + 1 + 2\right) = 1$$

$$\Rightarrow k\left(24 - \frac{10}{3}\right) = 1 \Rightarrow \frac{62}{3}k = 1$$

So $k = \frac{3}{62}$

6 b $\mathrm{F}(y)=\int_1^y \frac{3}{62}(t^2+2t+2)\,\mathrm{d}t=\left[\frac{3}{62}\left(\frac{t^3}{3}+t^2+2t\right)\right]_1^y$

$$=\frac{3}{62}\left(\frac{y^3}{3}+y^2+2y\right)-\frac{3}{62}\left(\frac{1}{3}+1+2\right)=\frac{y^3}{62}+\frac{3y^2}{62}+\frac{3y}{31}-\frac{5}{31}$$

So the cumulative distribution function is:

$$\mathrm{F}(y)=\begin{cases}0 & y<1\\ \frac{y^3}{62}+\frac{3y^2}{62}+\frac{3y}{31}-\frac{5}{31} & 1\leqslant y\leqslant 3\\ 1 & y>3\end{cases}$$

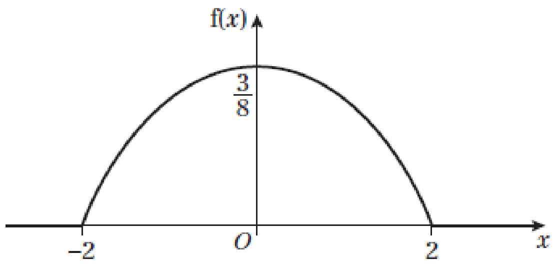
c $\mathrm{P}(Y\leqslant 2)=\mathrm{F}(2)=\frac{2^3}{62}+\frac{3\times 2^2}{62}+\frac{6}{31}-\frac{5}{31}=\frac{4+6+6-5}{31}=\frac{11}{31}$

Alternatively, the probability can be derived from the probability density function as follows:

$$\mathrm{P}(Y\leqslant 2)=\int_1^2 \frac{3}{62}(y^2+2y+2)\,\mathrm{d}y=\left[\frac{3}{62}\left(\frac{y^3}{3}+y^2+2y\right)\right]_1^2$$

$$=\frac{3}{62}\left(\frac{2^3}{3}+2^2+4\right)-\frac{3}{62}\left(\frac{1}{3}+1+2\right)=\frac{11}{31}$$

7 a The graph of the probability density function is a quadratic with a negative x^2 coefficient between (–2, 0) and (2, 0), with a maximum at (0, 0.375); otherwise it is 0. The sketch of the graph is:



b The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when $x = 0$. So the mode is 0.

7 c $\mathrm{F}(x)=\int_{-2}^{x}\frac{3}{32}(4-t^2)\,\mathrm{d}t=\left[\frac{12t}{32}-\frac{t^3}{32}\right]_{-2}^{x}$

$=\left(\frac{12x}{32}-\frac{x^3}{32}\right)-\left(-\frac{24}{32}+\frac{8}{32}\right)=\frac{12x}{32}-\frac{x^3}{32}+\frac{1}{2}$

So the cumulative distribution function is:

$$\mathrm{F}(x)=\begin{cases}0 & x<-2\\ \frac{12x}{32}-\frac{x^3}{32}+\frac{1}{2} & -2\leqslant x\leqslant 2\\ 1 & x>2\end{cases}$$

d $\mathrm{P}(0.5<X<1.5)=\mathrm{P}(X<1.5)-\mathrm{P}(X<0.5)=\mathrm{F}(1.5)-\mathrm{F}(0.5)$

$=\left(\frac{18}{32}-\frac{1.5^3}{32}+\frac{1}{2}\right)-\left(\frac{6}{32}-\frac{0.5^3}{32}+\frac{1}{2}\right)=\frac{12}{32}-\frac{26}{256}=\frac{96-26}{256}=\frac{35}{128}=0.273\text{ (3 s.f.)}$

8 a $\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_0^1\frac{x}{3}\mathrm{d}x+\int_1^2\frac{2x^3}{7}\,\mathrm{d}x$

$=\left[\frac{x^2}{6}\right]_0^1+\left[\frac{2x^4}{28}\right]_1^2=\frac{1}{6}+\left(\frac{32}{28}-\frac{2}{28}\right)$

$=\frac{1}{6}+\frac{15}{14}=\frac{7}{42}+\frac{45}{42}=\frac{52}{42}=\frac{26}{21}=1.238\text{ (4 s.f.)}$

b If $x\leqslant 0$, $\mathrm{F}(x)=0$ so $\mathrm{F}(0)=0$

If $0\leqslant x<1$

$\mathrm{F}(x)=\mathrm{F}(0)+\int_0^x\frac{1}{3}\mathrm{d}t=\left[\frac{t}{3}\right]_0^x=\frac{x}{3}$

So $\mathrm{F}(1)=\frac{1}{3}$

If $1\leqslant x\leqslant 2$

$\mathrm{F}(x)=\mathrm{F}(1)+\int_1^x\frac{2}{7}t^2\,\mathrm{d}t=\frac{1}{3}+\left[\frac{2t^3}{21}\right]_1^x=\frac{1}{3}+\frac{2x^3}{21}-\frac{2}{21}=\frac{2x^3}{21}+\frac{5}{21}$

So the cumulative distribution function is:

$$\mathrm{F}(x)=\begin{cases}0 & x<0\\ \frac{x}{3} & 0\leqslant x<1\\ \frac{2x^3}{21}+\frac{5}{21} & 1\leqslant x\leqslant 2\\ 1 & x>2\end{cases}$$

8 c i $F(1)=\frac{1}{3}$ therefore the median lies in the interval $1\leqslant x\leqslant 2$, so

$$F(m)=0.5,\ \text{so}\ \frac{2m^3}{21}+\frac{5}{21}=0.5$$
$$\Rightarrow 4m^3+10=21$$
$$\Rightarrow m^3=\frac{11}{4}=2.75$$
$$\Rightarrow m=1.401\ (4\text{ s.f.})$$

ii $F(P_{15})=\frac{15}{100}$; so as $F(P_{15})\leqslant\frac{1}{3}$, P_{15} is in the interval $0\leqslant x<1$

Therefore $\frac{1}{3}P_{15}=\frac{15}{100}\Rightarrow P_{15}=\frac{45}{100}=0.45$

9 $F(1)=0\Rightarrow 0.05a-b=0\Rightarrow b=0.05a$

$F(2)=1\Rightarrow 0.05a^2-b=1$

$\Rightarrow 0.05(a^2-a)=1$ substituting for b

$\Rightarrow a^2-a-20=0$

$\Rightarrow (a+4)(a-5)=0$ factoring

Since a is positive, $a=5$

So $b=0.05a=\frac{1}{20}\times 5=\frac{1}{4}=0.25$

10 If $F(x)$ is a cumulative distribution function, then the probability distribution function $f(x)$ is found by:

$$\frac{d}{dx}F(x)=f(x)$$
$$\frac{d}{dx}\left(\frac{1}{5}(16x-x^2-55)\right)=\frac{1}{5}(16-2x)$$

So

$$f(x)=\begin{cases}\frac{2}{5}(8-x) & 5\leqslant x\leqslant 10\\ 0 & \text{otherwise}\end{cases}$$

But this cannot be a probability distribution function as $f(x)<0$ for $8<x\leqslant 10$. So $F(x)$ cannot be a cumulative distribution function.

11 a The area under the probability density function graph must be 1, so:

$$\int_1^3 kx-k\,dx=1$$
$$\Rightarrow k\left[\frac{x^2}{2}-x\right]_1^3=1$$
$$\Rightarrow k\left(\frac{9}{2}-3-\frac{1}{2}+1\right)=1$$
$$\Rightarrow 2k=1\Rightarrow k=\frac{1}{2}$$

11 b $\mathrm{E}(X) = \int x\,\mathrm{f}(x)\mathrm{d}x = \int_1^3 \frac{1}{2}(x^2 - x)\,\mathrm{d}x = \left[\frac{x^3}{6} - \frac{x^2}{4}\right]_1^3$

$= \frac{27}{6} - \frac{9}{4} - \frac{1}{6} + \frac{1}{4} = \frac{13}{3} - 2 = \frac{7}{3} = 2.33$ (3 s.f.)

c $\mathrm{F}(x) = \int_1^x \frac{1}{2}(t-1)\,\mathrm{d}t = \left[\frac{t^2}{4} - \frac{t}{2}\right]_1^x = \frac{x^2}{4} - \frac{x}{2} - \frac{1}{4}$

So the cumulative distribution function is:

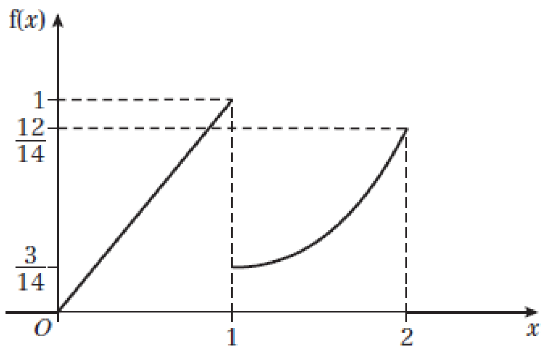
$$\mathrm{F}(x) = \begin{cases} 0 & x < 1 \\ \frac{x^2}{4} - \frac{x}{2} + \frac{1}{4} & 1 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

d $\mathrm{F}(2.4) = \frac{2.4^2}{4} - 1.2 + 0.25 = 0.49$

$\mathrm{F}(2.5) = \frac{2.5^2}{4} - 1.25 + 0.25 = 0.5625$

Since $\mathrm{F}(2.4) < 0.5 < \mathrm{F}(2.5)$, the median, m, when $\mathrm{F}(m) = 0.5$ lies between 2.4 and 2.5.

12 a The graph is a straight line from (0, 0) to (1, 1); part of a quadratic with a positive x^2 coefficient from $\left(1, \frac{3}{14}\right)$ to $\left(2, \frac{12}{14}\right)$; and otherwise 0. The sketch of the graph is:



b The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when $x = 1$. So the mode is 1.

c $\mathrm{E}(X) = \int x\,\mathrm{f}(x)\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x + \int_1^2 \frac{3x^3}{14}\mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{3x^4}{56}\right]_1^2$

$= \frac{1}{3} + \frac{48}{56} - \frac{3}{56} = \frac{56}{168} + \frac{144}{168} - \frac{9}{168} = \frac{191}{168}$

$\mathrm{E}(2X) = 2\mathrm{E}(X) = 2 \times \frac{191}{168} = \frac{191}{84}$

12 d $\text{Var}(X)=\int x^2\,\text{f}(x)\text{d}x-(\text{E}(X))^2=\int_0^1 x^3\text{d}x+\int_1^2\frac{3x^4}{14}\text{d}x-\left(\frac{191}{168}\right)^2$

$$=\left[\frac{x^4}{4}\right]_0^1+\left[\frac{3x^5}{70}\right]_1^2-\left(\frac{191}{168}\right)^2$$

$$=\frac{1}{4}+\frac{96}{70}-\frac{3}{70}-\left(\frac{191}{168}\right)^2=0.2860\text{ (4 s.f.)}$$

$\text{Var}(2X+1)=2^2\,\text{Var}(X)=4\times0.28602=1.14$ (3 s.f.)

e If $x\leqslant 0$, $\text{F}(x)=0$ so $\text{F}(0)=0$

If $0\leqslant x\leqslant 1$

$$\text{F}(x)=\text{F}(0)+\int_0^x t\,\text{d}t=\left[\frac{t^2}{2}\right]_0^x=\frac{x^2}{2}$$

So $\text{F}(1)=\frac{1}{2}$

If $1<x\leqslant 2$

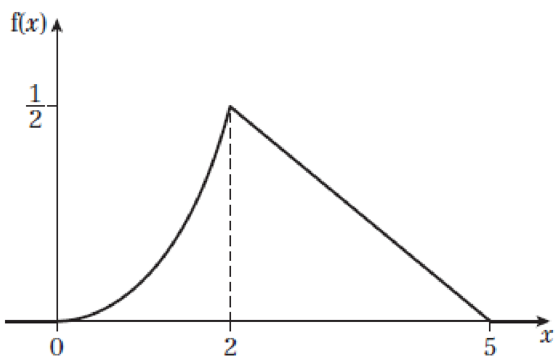
$$\text{F}(x)=\text{F}(1)+\int_1^x\frac{3t^2}{14}\text{d}t=\frac{1}{2}+\left[\frac{3t^3}{42}\right]_1^x=\frac{1}{2}+\frac{3x^3}{42}-\frac{3}{42}=\frac{3x^3}{42}+\frac{18}{42}=\frac{3x^3}{42}+\frac{3}{7}$$

So the cumulative distribution function is:

$$\text{F}(x)=\begin{cases}0 & x<0\\ \dfrac{x^2}{2} & 0\leqslant x\leqslant 1\\ \dfrac{x^3}{14}+\dfrac{3}{7} & 1<x\leqslant 2\\ 1 & x>2\end{cases}$$

f In deriving the cumulative distribution function in part **e**, it was shown that $\text{F}(1)=0.5$.
So the median is 1.

13 a The graph is part of a cubic with a positive x^3 coefficient from $(0,0)$ to $(2,0.5)$; a straight line from (2, 0.5) to (5, 0); and otherwise 0. The sketch of the graph is:



b The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when $x=2$. So the mode is 2.

13 c Using the sketch, $P(X > 2)$ = area of triangle with coordinates (2, 0), (2, 0.5) and (5, 0)

So $P(X>2)=\frac{1}{2}\times 3\times\frac{1}{2}=0.75$

d If $x \leqslant 0$, $F(x)=0$ so $F(0)=0$

If $0 \leqslant x < 2$

$$F(x)=F(0)+\int_0^x \frac{t^3}{16}\,dt=\left[\frac{t^4}{64}\right]_0^x=\frac{x^4}{64}$$

So $F(2)=\frac{2^4}{64}=\frac{1}{4}$

If $2 \leqslant x \leqslant 5$

$$F(x)=F(2)+\int_2^x \frac{5-t}{6}\,dt=\frac{1}{4}+\left[\frac{5t}{6}-\frac{t^2}{12}\right]_2^x=\frac{1}{4}+\frac{5x}{6}-\frac{x^2}{12}-\frac{10}{6}+\frac{4}{12}=\frac{1}{12}(10x-x^2-13)$$

So the cumulative distribution function is:

$$F(x)=\begin{cases} 0 & x<0 \\ \frac{x^4}{64} & 0 \leqslant x < 2 \\ \frac{10x-x^2-13}{12} & 2 \leqslant x \leqslant 5 \\ 1 & x>5 \end{cases}$$

e As F(2) = 0.25 (from part **d**), the median, m, lies in the range $2 \leqslant m \leqslant 5$

$F(m)=0.5 \Rightarrow \frac{10m-m^2-13}{12}=0.5 \Rightarrow m^2-10m+19=0$

So $m=\frac{10\pm\sqrt{100-76}}{2}=5\pm\sqrt{6}$

As $5+\sqrt{6}$ is outside the range, $m=5-\sqrt{6}=2.55$ (3 s.f.)

14 a $\frac{d}{dx}F(x)=f(x)$, so

$$f(x)=\frac{d}{dx}\left(\frac{1}{81}(-2x^3+15x^2-44)\right)=\frac{1}{81}(-6x^2+30x)=-\frac{2}{27}x^2+\frac{10}{27}x=\frac{2x}{27}(5-x)$$

So $f(x)=\begin{cases} \frac{2x}{27}(5-x) & 2 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$

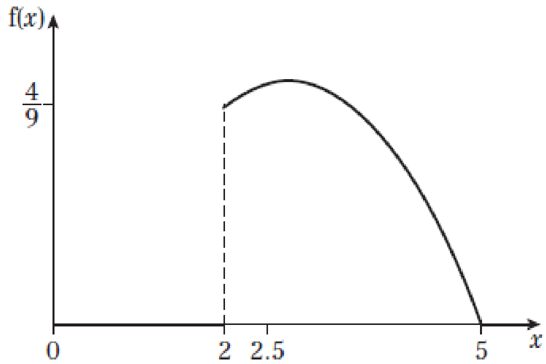
14 b To find the value of x when f(x) is a maximum, solve $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x)=0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{2}{27}x^2+\frac{10}{27}x\right)=-\frac{4}{27}x+\frac{10}{27}$$

So the mode occurs when $-4x+10=0 \Rightarrow x=2.5$

Note that f(x) is clearly a maximum at this point as the function is a negative quadratic.

c The graph is part of a quadratic with a negative x^2 coefficient from $\left(2,\frac{4}{9}\right)$ to $(5,0)$; and is otherwise 0. The quadratic has a maximum at $\left(2.5,\frac{25}{54}\right)$. The sketch of the graph is:



d $\mu=\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_2^5 -\frac{2}{27}x^3+\frac{10}{27}x^2\mathrm{d}x$

$$=\frac{2}{27}\left[-\frac{1}{4}x^4+\frac{5}{3}x^3\right]_2^5=\frac{2}{27}\left(-\frac{625}{4}+\frac{625}{3}+4-\frac{40}{3}\right)=\frac{2}{27}\left(\frac{-1875+2500+48-160}{12}\right)$$

$$=\frac{2}{27}\times\frac{513}{12}=\frac{2\times19}{12}=\frac{19}{6}$$

e $\mathrm{F}(\mu)=\mathrm{F}\left(\frac{19}{6}\right)=\frac{1}{81}\left(-2\left(\frac{19}{6}\right)^3+15\left(\frac{19}{6}\right)^2-44\right)=0.5297$ (4 s.f.)

f $\mathrm{F}(2.5)=\frac{1}{81}\left(-2(2.5)^3+15(2.5)^2-44\right)=0.2284$ (4 s.f.)

If m is the median, F(m) =0.5, so, as 0.2284 < 0.5 < 0.5297, for this distribution, mode < median < mean.

15 a As F(x) is a cumulative distribution function, $\mathrm{F}(5)=1$

So $k(35\times5-2\times5^2)=1\Rightarrow125k=1\Rightarrow k=\frac{1}{125}$

15 b $F(m) = 0.5$, so $\frac{1}{125}(35m - 2m^2) = 0.5$

$\Rightarrow 4m^2 - 70m + 125 = 0$

$\Rightarrow m = \frac{70 \pm \sqrt{4900 - 2000}}{8} = \frac{70 \pm 53.8516\ldots}{8}$

As $\frac{70 + 53.8516\ldots}{8} > 5$, this cannot be a solution as the median lies between 0 and 5

So the solution is $m = \frac{70 - 53.8516\ldots}{8} = 2.02$ (3 s.f.)

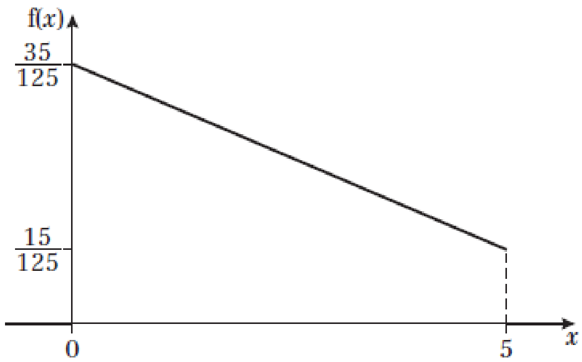
c $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$, so

$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{125}(35x - 2x^2)\right) = \frac{1}{125}(35 - 4x)$

So $f(x) = \begin{cases} \frac{1}{125}(35 - 4x) & 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$

d The graph is a straight line from $\left(0, \frac{35}{125}\right)$ to $\left(5, \frac{15}{125}\right)$ and is otherwise 0.

The sketch of the graph is:



e The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when $x = 0$. So the mode is 0.

f $E(X) = \int x f(x)\mathrm{d}x = \int_0^5 \frac{1}{125}(35x - 4x^2)\mathrm{d}x = \frac{1}{125}\left[\frac{35}{2}x^2 - \frac{4}{3}x^3\right]_0^5$

$= \frac{1}{125}\left(\frac{35 \times 5^2}{2} - \frac{4 \times 5^3}{3}\right) = \frac{7}{2} - \frac{4}{3} = \frac{21}{6} - \frac{8}{6} = \frac{13}{6} = 2.17$ (3 s.f.)

g P_5 is such that $F(a) = 0.05$

So $\frac{1}{125}(35a - 2a^2) = 0.05 \Rightarrow 2a^2 - 35a + 6.25 = 0$

$a = \frac{35 \pm \sqrt{1225 - 50}}{4} = \frac{35 \pm 34.27827\ldots}{4} = 0.180$ or 17.3 (3 s.f.)

So 5th percentile = 0.180 (3 s.f.) as other answer outside range

16 The area under the probability density function graph must be 1, so:

$$\int_0^2 ax + b\,\mathrm{d}x = \left[\frac{1}{2}ax^2 + bx\right]_0^2 = 1 \Rightarrow 2a + 2b = 1 \qquad (1)$$

$$\mathrm{E}(X) = \int x\,\mathrm{f}(x)\mathrm{d}x = \int_0^2 ax^2 + bx\,\mathrm{d}x = \left[\frac{1}{3}ax^3 + \frac{1}{2}bx^2\right]_0^2 = \frac{9}{8} \Rightarrow \frac{8}{3}a + 2b = \frac{9}{8} \qquad (2)$$

Subtracting equation (2) from equation (1) gives:

$\frac{2}{3}a = \frac{1}{8} \Rightarrow a = \frac{3}{16}$

Substituting for a in equation (1) gives:

$\frac{6}{16} + 2b = 1 \Rightarrow b = \frac{5}{16}$

17 a The area under the probability density function graph must be 1, so:

$$\int_{-1}^0 k(x+1)^3\,\mathrm{d}x = \left[\frac{k(x+1)^4}{4}\right]_{-1}^0 = 1$$

$\Rightarrow \frac{k}{4} = 1 \Rightarrow k = 4$

b $\mathrm{E}(X) = \int x\,\mathrm{f}(x)\mathrm{d}x = \int_{-1}^0 4x(x+1)^3\,\mathrm{d}x = \int_{-1}^0 4x^4 + 12x^3 + 12x^2 + 4x\mathrm{d}x$

$= \left[\frac{4}{5}x^5 + 3x^4 + 4x^3 + 2x^2\right]_{-1}^0$

$= \frac{4}{5} - 3 + 4 - 2 = -0.2$

c $\mathrm{F}(x) = \int 4(x+1)^3\,\mathrm{d}x = (x+1)^4 + c$

$\mathrm{F}(-1) = 0 \Rightarrow c = 0$

So the cumulative distribution function is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < -1 \\ (x+1)^4 & -1 \leqslant x \leqslant 0 \\ 1 & x > 0 \end{cases}$$

d $\mathrm{F}(m) = (m+1)^4 = 0.5$

$\Rightarrow m + 1 = 0.8409$

$\Rightarrow m = -0.159$ (3 s.f.)

18 a $\mathrm{F}(t) = \int \frac{1}{72}(6-t)^2\,\mathrm{d}t = -\frac{1}{216}(6-t)^3 + c$

$\mathrm{F}(0) = 0 \Rightarrow c = 1$

So the cumulative distribution function is:

$$\mathrm{F}(t) = \begin{cases} 0 & t < 0 \\ 1 - \dfrac{(6-t)^3}{216} & 0 \leqslant t \leqslant 6 \\ 1 & t > 6 \end{cases}$$

b $\mathrm{F}(m) = 1 - \frac{(6-m)^3}{216} = 0.5$

$\Rightarrow (6-m)^3 = 108$

$\Rightarrow 6 - m = 4.7622\ldots$

$\Rightarrow m = 1.24$ hours (3 s.f.)

c $\mathrm{E}(T) = \int t\,\mathrm{f}(t)\mathrm{d}t = \int_0^6 \frac{t}{72}(6-t)^2\,\mathrm{d}t = \int_0^6 \frac{t}{2} - \frac{t^2}{6} + \frac{t^3}{72}\,\mathrm{d}t$

$= \left[\frac{t^2}{4} - \frac{t^3}{18} + \frac{t^4}{288}\right]_0^6 = \frac{36}{4} - \frac{216}{18} + \frac{1296}{288} = 9 - 12 + 4.5 = 1.5$ hours

19 $\mathrm{F}(x) = \int_1^x \frac{2}{(2t-1)\ln 5}\,\mathrm{d}t = \left[\frac{\ln(2t-1)}{\ln 5}\right]_1^x = \frac{\ln(2x-1)}{\ln 5}$

So the cumulative distribution function is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < 1 \\ \dfrac{\ln(2x-1)}{\ln 5} & 1 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

20 a The area under the probability density function graph must be 1, so:

$\int_0^1 kx\sin(\pi x)\,\mathrm{d}x = 1$

$\Rightarrow \left[-kx\frac{\cos(\pi x)}{\pi}\right]_0^1 - \int_0^1 -\frac{k\cos(\pi x)}{\pi}\,\mathrm{d}x = 1$ using integration by parts

$\Rightarrow \left[-kx\frac{\cos(\pi x)}{\pi} + \frac{k\sin(\pi x)}{\pi^2}\right]_0^1 = 1$

$\Rightarrow \frac{k}{\pi} = 1$

$\Rightarrow k = \pi$

20 b $\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_0^1 \pi x^2\sin(\pi x)\mathrm{d}x$

$=\left[-x^2\cos(\pi x)\right]_0^1+\int_0^1 2x\cos(\pi x)\mathrm{d}x$ using integration by parts

$=\left[-x^2\cos(\pi x)+2x\frac{\sin(\pi x)}{\pi}\right]_0^1-\int_0^1 2\frac{\sin(\pi x)}{\pi}\mathrm{d}x$ using integration by parts again

$=\left[-x^2\cos(\pi x)+2x\frac{\sin(\pi x)}{\pi}+\frac{2\cos(\pi x)}{\pi^2}\right]_0^1$

$=1-\frac{2}{\pi^2}-\frac{2}{\pi^2}=0.5947\simeq 59\%$

21 a The area under the probability density function graph must be 1, so:

$\int_0^1 k\mathrm{d}x+\int_1^2\frac{k}{x^2}\mathrm{d}x=1$

$\Rightarrow [kx]_0^1+\left[-\frac{k}{x}\right]_1^2=1$

$\Rightarrow k-\frac{k}{2}+k=1$

$\Rightarrow \frac{3k}{2}=1$

$\Rightarrow k=\frac{2}{3}$

b $\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_0^1\frac{2x}{3}\mathrm{d}x+\int_1^2\frac{2}{3x}\mathrm{d}x$

$=\left[\frac{x}{3}\right]_0^1+\left[\frac{2\ln x}{3}\right]_1^2=\frac{1}{3}+\frac{2}{3}\ln 2=0.79543\ldots=0.795\text{ (3 s.f.)}$

c $\mathrm{E}(X^2)=\int x^2\,\mathrm{f}(x)\mathrm{d}x=\int_0^1\frac{2x^2}{3}\mathrm{d}x+\int_1^2\frac{2}{3}\mathrm{d}x$

$=\left[\frac{2x^3}{9}\right]_0^1+\left[\frac{2x}{3}\right]_1^2=\frac{2}{9}+\frac{2}{3}=\frac{8}{9}$

So $\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2=\frac{8}{9}-(0.7954)^2=0.256\text{ (3 s.f.)}$

Challenge

a $\mathrm{E}(X)=\int_0^\infty x\mathrm{f}(x)\mathrm{d}x=\int_0^\infty x\lambda\mathrm{e}^{-\lambda x}\mathrm{d}x$

$=\left[x(\lambda\mathrm{e}^{-\lambda x})\right]_0^\infty-\int_0^\infty -\mathrm{e}^{-\lambda x}\mathrm{d}x$ using integration by parts

$=0+\int_0^\infty \mathrm{e}^{-\lambda x}\mathrm{d}x=\left[-\dfrac{\mathrm{e}^{-\lambda x}}{\lambda}\right]_0^\infty=\dfrac{1}{\lambda}$

$\mathrm{E}(X^2)=\int_0^\infty x^2\mathrm{f}(x)\mathrm{d}x=\int_0^\infty x^2(\lambda\mathrm{e}^{-\lambda x})\mathrm{d}x$

$=\left[x^2(\lambda\mathrm{e}^{-\lambda x})\right]_0^\infty-\int_0^\infty -2x\mathrm{e}^{-\lambda x}\mathrm{d}x$ using integration by parts

$=0+2\int_0^\infty x\mathrm{e}^{-\lambda x}\mathrm{d}x=\dfrac{2}{\lambda}\mathrm{E}(X)=\dfrac{2}{\lambda^2}$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2=\dfrac{2}{\lambda^2}-\left(\dfrac{1}{\lambda}\right)^2=\dfrac{1}{\lambda^2}$

b $\mathrm{P}(X>a)=1-\mathrm{P}(X<a)=1-\int_0^a \lambda\mathrm{e}^{-\lambda x}\mathrm{d}x$

$=1-\left[-\mathrm{e}^{-\lambda x}\right]_0^a=1-(-\mathrm{e}^{-\lambda a}+1)=\mathrm{e}^{-\lambda a}$

Similarly, $\mathrm{P}(X>b)=\mathrm{e}^{-\lambda b}$ and $\mathrm{P}(X>a+b)=\mathrm{e}^{-\lambda(a+b)}=\mathrm{e}^{-\lambda a}\times\mathrm{e}^{-\lambda b}$

Hence, $\mathrm{P}(X>a+b\mid X>a)=\dfrac{\mathrm{P}(X>a+b)}{\mathrm{P}(X>a)}=\dfrac{\mathrm{e}^{-\lambda a}\times\mathrm{e}^{-\lambda b}}{\mathrm{e}^{-\lambda a}}=\mathrm{e}^{-\lambda b}=\mathrm{P}(X>b)$