Chapter Review 3

1 a The distribution is binomial, $\mathrm{B}(100, 0.40)$.

The binomial distribution can be approximated by the normal distribution when n is large (> 50) and p is close to 0.5. Here n = 100 and p = 0.4 so both of these conditions are satisfied.

b $\mu = np = 100 \times 0.4 = 40$ and $\sigma = \sqrt{np(1-p)} = \sqrt{40 \times 0.6} = \sqrt{24} = 4.899$ (4 s.f.)

c Let $Y \sim \mathrm{N}(40, 24)$, then

$\mathrm{P}(X \geqslant 50) \approx \mathrm{P}(Y \geqslant 49.5) = 0.02623... = 0.0262$ (4 d.p.)

2 a $\mathrm{P}(X = 65) = \binom{120}{65} \times 0.46^{65} \times 0.54^{55} = 0.01467$ (4 s.f.) $= 0.0147$ (4 d.p.)

b The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: n = 120 is large (> 50) and p = 0.46 is close to 0.5.

$\mu = np = 120 \times 0.46 = 55.2$ and $\sigma = \sqrt{np(1-p)} = \sqrt{55.2 \times 0.54} = \sqrt{29.808} = 5.460$ (4 s.f.)

c Let $Y \sim \mathrm{N}(55.2, 5.46^2)$

Using the normal cumulative distribution function, $\mathrm{P}(X = 65) \approx \mathrm{P}(64.5 < Y < 65.5) = 0.01463...$

Percentage error $= \dfrac{0.01467 - 0.01463}{0.01467} \times 100 = \dfrac{0.00004}{0.01467} \times 100 = 0.27\%$

3 a The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: n = 300 is large (> 50) and p = 0.6 is close to 0.5.

b $\mu = np = 300 \times 0.6 = 180$ and $\sigma = \sqrt{np(1-p)} = \sqrt{180 \times 0.4} = \sqrt{72} = 8.485$ (4 s.f.)

So let $X \sim \mathrm{N}(180, 8.485^2)$.

$\mathrm{P}(150 < Y \leqslant 180) \approx \mathrm{P}(150.5 < X < 180.5) = 0.52324... = 0.5232$ (4 d.p.)

c Using the inverse normal distribution, $\mathrm{P}(X < a) = 0.05 \Rightarrow a = 166.04$

So $\mathrm{P}(X < 166.5) > 0.05$ and $\mathrm{P}(X < 165.5) < 0.05$

So $\mathrm{P}(Y < 167) \approx \mathrm{P}(X < 166.5) > 0.05$ and $\mathrm{P}(Y < 166) \approx \mathrm{P}(X < 165.5) < 0.05$

So the largest value of y such that $\mathrm{P}(Y < y) < 0.05$ is y = 166.

4 The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: n = 80 is large (> 50) and p = 0.4 is close to 0.5.

$\mu = np = 80 \times 0.4 = 32$ and $\sigma = \sqrt{np(1-p)} = \sqrt{32 \times 0.6} = \sqrt{19.2} = 4.382$ (4 s.f.)

So let $Y \sim \mathrm{N}(32, 4.382^2)$

$\mathrm{P}(X > 30) \approx \mathrm{P}(Y > 30.5) = 0.63394... = 0.6339$ (4 d.p.)

5 a Use the binomial distribution $X \sim \mathrm{B}(20, 0.55)$

Using the binomial cumulative function,

$\mathrm{P}(X > 10) = 1 - \mathrm{P}(X \leqslant 10) = 1 - 0.40863\ldots = 0.5914$ (4 d.p.)

5 b A normal approximation is valid since $n = 200$ is large (> 50) and $p = 0.55$ is close to 0.5.

$\mu = np = 200 \times 0.55 = 110$ and $\sigma = \sqrt{np(1-p)} = \sqrt{110 \times 0.45} = \sqrt{49.5} = 7.036$ (4 s.f.)

So let $Y \sim \mathrm{N}(110, 7.036^2)$

$\mathrm{P}(X \leqslant 95) \approx \mathrm{P}(Y < 95.5) = 0.01965... = 0.0197$ (4 d.p.)

c It seems unlikely that the company's claim is correct: the chance of only 95 (or fewer) seedlings producing apples from a sample of 200 seedlings would be less than 2%.

6 a Use the binomial distribution $X \sim \mathrm{B}(25, 0.52)$

Using the binomial cumulative distribution function,

$\mathrm{P}(X > 12) = 1 - \mathrm{P}(X \leqslant 12) = 1 - 0.41992... = 0.5801$ (4 d.p.)

b A normal approximation is valid since $n = 300$ is large (> 50) and $p = 0.52$ is close to 0.5.

$\mu = np = 300 \times 0.52 = 156$ and $\sigma = \sqrt{np(1-p)} = \sqrt{156 \times 0.48} = \sqrt{74.88} = 8.653$ (4 s.f.)

So let $Y \sim \mathrm{N}(156, 8.653^2)$

$\mathrm{P}(X \geqslant 170) \approx \mathrm{P}(Y > 169.5) = 0.05936... = 0.0594$ (4 d.p.)

c There is a less than 6% chance that 170 people out of 300 would be cured if the remedy has a 52% success rate, therefore it is likely that the doctor has understated the actual cure rate.

7 a Let X represent the number of sixes obtained.

$X \sim \mathrm{B}\left(48, \frac{1}{6}\right)$

Use the approximation $Y \sim \mathrm{Po}(8)$

$\mathrm{P}(X \leq 10) = 0.8159$

b Let X represent the number of sixes obtained.

$X \sim \mathrm{B}\left(120, \frac{1}{6}\right)$

Use the approximation $Y \sim \mathrm{N}\left(20, \frac{50}{3}\right)$

$$\begin{aligned}
\mathrm{P}(X \geq 25) &= \mathrm{P}(Y > 24.5) \\
&= \mathrm{P}\left(Z > \frac{24.5 - 20}{\sqrt{\frac{50}{3}}}\right) \\
&= 1 - \mathrm{P}\left(Z < \frac{24.5 - 20}{\sqrt{\frac{50}{3}}}\right) \\
&= 1 - \mathrm{P}(Z < 1.102...) \\
&= 1 - 0.8648 \\
&= 0.1352
\end{aligned}$$

8 Let X = number of heads in 60 tosses of a fair coin, so $X \sim \text{B}(60, 0.5)$.
Since $p = 0.5$ and 60 is large, X can be approximated by the normal distribution $Y \sim \text{N}(\mu, \sigma^2)$,
where $\mu = 60 \times 0.5 = 30$ and $\sigma = \sqrt{60 \times 0.5 \times 0.5} = \sqrt{15}$
So $Y \sim \text{N}(30, 15)$
$\text{P}(X < 25) \approx \text{P}(Y < 24.5) = 0.07779... = 0.0778$ (4 d.p.)

9 **a** Let X be the number of customers who bought a newspaper.
$X \sim \text{B}(100, 0.4)$
Use the approximation $Y \sim \text{N}(40, 24)$.

$$\begin{aligned}\text{P}(X \geq 50) &= \text{P}(Y > 49.5)\\ &= \text{P}\left(Z > \frac{49.5 - 40}{\sqrt{24}}\right)\\ &= 1 - \text{P}\left(Z < \frac{49.5 - 40}{\sqrt{24}}\right)\\ &= 1 - \text{P}(Z < 1.939..)\\ &= 1 - 0.9737\\ &= 0.02623\end{aligned}$$

b Let X be the number of customers who spend over \$10.
$X \sim \text{B}(100, 0.04)$
Use the approximation $Y \sim \text{Po}(4)$.

$$\begin{aligned}\text{P}(Y > 5) &= 1 - \text{P}(Y \leq 5)\\ &= 1 - 0.7851\\ &= 0.2149\end{aligned}$$

10 $X \sim \text{Po}(3.5)$

a $$\text{P}(X = 2) = \frac{e^{-3.5} 3.5^2}{2!} = 0.185 \text{ (3 s.f.)}$$

b $$\text{P}(X < 6) = e^{-3.5}\left(\frac{3.5^0}{0!} + \frac{3.5^1}{1!} + \frac{3.5^2}{2!} + \frac{3.5^3}{3!} + \frac{3.5^4}{4!} + \frac{3.5^5}{5!}\right) = 0.858 \text{ (3 s.f.)}$$

c $X \sim \text{Po}(35)$
Use the approximation $Y \sim \text{N}(35, 35)$
$\text{P}(X < 45) = \text{P}(Y < 44.5)$ (apply a continuity correction)

$$\begin{aligned}\text{P}(Y < 44.5) &= \text{P}\left(Z < \frac{44.5 - 35}{\sqrt{35}}\right)\\ &= \text{P}(Z < 1.605...)\\ &= 0.946 \text{ (3 s.f.)}\end{aligned}$$

11 Let the number of people who buy large chocolate bars purchased be X.
$X \sim B(80, 0.2)$

a Use the approximation $Y \sim N(16, 12.8)$.
$P(Y > 20.5) = 1 - P(Y \le 20.5)$ (apply continuity correction)
$= 1 - P(Y < 20.5)$
$= 1 - P\left(Z < \frac{20.5 - 16}{\sqrt{12.8}}\right)$
$= 1 - P(Z < 1.257...)$
$= 1 - 0.8957...$
$= 0.104$ (3 s.f.)

b Let the number of people who buy extra large chocolate bars purchased be X.
$X \sim B(150, 0.02)$
Use the approximation $Y \sim Po(3)$
$P(Y < 5) = 0.815$ (3 s.f.)

Challenge

Let X be the number of defective switches in a box of 500.
$X \sim B(500, 0.005)$

a $P(X < 2) = P(X = 0) + P(X = 1)$
$= {}^{500}C_0 (0.005)^0 (0.995)^{500} + {}^{500}C_1 (0.005)^1 (0.995)^{499}$
$= 0.2865$ (4 s.f.)

b $Y \sim Po(2.5)$
$P(Y < 2) = P(Y = 0) + P(Y = 1)$
$= \frac{e^{-2.5} 2.5^0}{0!} + \frac{e^{-2.5} 2.5^1}{1!}$
$= 0.2873$ (4 s.f.)

c $W \sim N(2.5, 2.4875)$
$P(W < 2) = P(W < 1.5)$ (apply continuity correction)
$P(W < 1.5) = P\left(Z < \frac{1.5 - 2.5}{\sqrt{2.4875}}\right)$
$= P(Z < -0.6340...)$
$= 1 - P(Z < 0.6340...)$
$= 1 - 0.7369...$
$= 0.2631$ (4 s.f.)

d For binomial to Poisson,
Percentage error $= \frac{0.2873 - 0.2865}{0.2865} \times 100 = 0.279\%$ (3 s.f.)
For binomial to normal,
Percentage error $= \frac{0.2865 - 0.2631}{0.2865} \times 100 = 8.17\%$ (3 s.f.)
So the Poisson is a more accurate approximation.