Poisson distributions 2C

1 a i The probability that there are exactly 4 requests for replacement light bulbs is P(X = 4).
As $X \sim \text{Po}(3)$

$$P(X=4)=\frac{e^{-3}\times 3^4}{4!}=0.1680 \text{ (4 d.p.)}$$

ii The probability that there are more than 5 requests is P(X > 5).
Find this from the tables using $\lambda = 3$

$$P(X>5)=1-P(X\leqslant 5)=1-0.9161=0.0839$$

b Let Y be the number of requests in a fortnight, so use the tables with $\lambda = 6$.

i As $Y \sim \text{Po}(6)$

$$P(Y=6)=\frac{e^{-6}\times 6^6}{6!}=0.1606 \text{ (4 d.p.)}$$

ii Use the tables with $\lambda = 6$

$$P(X\leqslant 4)=0.2851$$

2 a Weeds must grow independently of the presence of other weeds. They must grow at a constant average density so that the mean number in any area of the field is proportional to the area.

b Let X be the number of weeds in a random 4 m^2 area of the field. In this model, $X \sim \text{Po}(4\times 1.3)$, i.e. $X \sim \text{Po}(5.2)$, so $\lambda = 5.2$.

As the tables in the textbook do not give values of the Poisson cumulative distribution function for $\lambda = 5.2$ and the required probability $P(X\leqslant 2)$ must be found using a calculator.

$$P(X\leqslant 2)=0.1088$$

c Let Y be the number of weeds in a random 5 m^2 area of the field. In this model, $X \sim \text{Po}(5\times 1.3)$, i.e. $X \sim \text{Po}(6.5)$, so $\lambda = 6.5$ and the tables can be used to find the required value.

$$P(X>8)=1-P(X\leqslant 8)=1-0.7916=0.2084$$

3 a Detection occurs at a constant mean rate of 2.5. So a suitable model is to let X be the number of faulty components detected in a hour, with $X \sim \text{Po}(2.5)$.

b It is assumed that faulty components are found independently of each other and that the detection is evenly spread throughout each hour (so that the mean rate of detection in k hours is 2.5k for all positive values of k).

c As $X \sim \text{Po}(2.5)$

$$P(X=2)=\frac{e^{-2.5}\times 2.5^2}{2!}=0.2565 \text{ (4 d.p.)}$$

d Let Y be the number of faulty components detected in a 3-hour period, so $Y \sim \text{Po}(7.5)$.
Use the tables with $\lambda = 7.5$

$$P(X\geqslant 6)=1-P(X\leqslant 5)=1-0.2414=0.7586$$

3 e Let Z be the number of faulty components detected in a 4-hour period, so $Z \sim \text{Po}(10)$.
Use the tables with $\lambda = 10$
$P(X \geqslant 7) = 1 - P(X \leqslant 6) = 1 - 0.1301 = 0.8699$

4 a Let X be the number of telephone calls in a 20-minute interval, so $X \sim \text{Po}(5)$.

i $P(X = 4) = \dfrac{e^{-5} \times 5^4}{4!} = 0.1755$ (4 d.p.)

ii Use the tables with $\lambda = 5$
$P(X > 8) = 1 - P(X \leqslant 8) = 1 - 0.9319 = 0.0681$

b Let Y be the number of telephone calls in a 30-minute interval, so $Y \sim \text{Po}(7.5)$.

i Use the tables with $\lambda = 7.5$
$P(X \geqslant 6) = 1 - P(X \leqslant 5) = 1 - 0.2414 = 0.7586$

ii Use the tables with $\lambda = 7.5$
$P(X \leqslant 10) = 0.8622$

5 a Let X be the number of cars crossing in any given minute, so $X \sim \text{Po}(3)$, $\lambda = 3$.
$P(X > 5) = 1 - P(X \leqslant 5) = 1 - 0.9161 = 0.0839$

b Let Y be the number of cars crossing in any given 2-minute period, so $Y \sim \text{Po}(6)$, $\lambda = 6$.
$P(X \leqslant 3) = 0.1512$

6 Let X be the number of customers arriving for breakfast between 10:00 am and 10:20 am.
As 20 minutes is 5×4 minutes, the model is $X \sim \text{Po}(5)$.

a Use the tables with $\lambda = 5$
$P(X \leqslant 2) = 0.1247$

b Use the tables with $\lambda = 5$
$P(X > 10) = 1 - P(X \leqslant 10) = 1 - 0.9863 = 0.0137$

7 a Let X be the number of houses the estate agent sells in a week, so $X \sim \text{Po}(1.8)$. As $\lambda = 1.8$, all answers must be found using a calculator.

i $P(X = 0) = \dfrac{e^{-1.8} \times 1.8^0}{0!} = 0.1653$ (4 d.p.)

ii $P(X = 3) = \dfrac{e^{-1.8} \times 1.8^3}{3!} = 0.1607$ (4 d.p.)

iii $P(X \geqslant 3) = 1 - P(X \leqslant 2) = 1 - 0.7306 = 0.2694$ (4 d.p.)

7 b Let Y be the number of weeks, over a period of 4 weeks, in which the estate agent meets her target. As the probability that the agent meets her target is $P(X \geqslant 3) = 0.2694$ (from part **aiii**), the model is $Y \sim B(4, 0.2694)$.

$P(Y = 1) = \binom{4}{1}(0.2694)^1(1-0.2694)^3 = 0.4202$ (4 d.p.)

8 a Let X be the number of patients arriving during a 30-minute period, so $X \sim Po(2.5)$.

i $P(X = 4) = \frac{e^{-2.5} \times 2.5^4}{4!} = 0.1336$ (4 d.p.)

ii Use the tables with $\lambda = 2.5$

$P(X \geqslant 3) = 1 - P(X \leqslant 2) = 1 - 0.5438 = 0.4562$

b If the next patient arrives before 11:15 am then there must be at least one patient in the 15-minute period between 11:00 am and 11:15 am.
Let Y be the number of patients arriving during a 15-minute period, so $Y \sim Po(1.25)$. As $\lambda = 1.25$, the solution must be found using a calculator.

$P(X \geqslant 1) = 1 - P(X = 0) = 1 - \frac{e^{-1.25} \times 1.25^0}{0!} = 1 - 0.2865 = 0.7135$ (4 d.p.)

9 a Let X be the number of times the elevator breaks down in one week, $Y \sim Po(0.75)$. As $\lambda = 0.75$, the solutions must be found using a calculator.

i $P(X \geqslant 1) = 1 - P(X = 0) = 1 - \frac{e^{-0.75} \times 0.75^0}{0!} = 1 - 0.4724 = 0.5276$ (4 d.p.)

ii $P(X = 2) = \frac{e^{-0.75} \times 0.75^2}{2!} = 0.1329$ (4 d.p.)

b If the elevator breaking down can be modelled using a Poisson distribution, then each breakdown occurs independently of any previous history. So the probability of at least one breakdown in the next week will be $P(X \geqslant 1) = 0.5276$, as given in part **ai**.

10 a Let X be the number of flaws in a 50 m length of material, so $X \sim Po(1.5)$.

$P(X = 3) = \frac{e^{-1.5} \times 1.5^3}{3!} = 0.1255$ (4 d.p.)

b Let Y be the number of flaws in a 200 m length of material, so $Y \sim Po(6)$.
Use the tables with $\lambda = 6$

$P(X < 4) = P(X \leqslant 3) = 0.1512$

10 c Let A be the number of rolls in a random sample of 5 which have fewer than 4 flaws. As $P(X < 4) = 0.1512$ (from part **b**), the model is $A \sim B(5, 0.1512)$.

$$P(A \geqslant 2) = 1 - \binom{5}{1}(0.1512)^1(1-0.1512)^4 - \binom{5}{0}(0.1512)^0(1-0.1512)^5$$
$$= 0.1670$$

11 a Let X be the number of chocolate chips in a biscuit, so $X \sim Po(5)$.
Use the tables with $\lambda = 5$
$P(X < 3) = P(X \leqslant 2) = 0.1247$

b Let Y be the number of biscuits in a pack of 6 which contain fewer than 3 chocolate chips. As $P(X < 3) = 0.1247$ (from part **a**), the model is $Y \sim B(6, 0.1247)$.

$$P(Y = 3) = \binom{6}{3}(0.1247)^3(1-0.1247)^3 = 0.0260$$

12 a Let X be the number of requests for buses on a Sunday in summer, so $X \sim Po(5)$.
Use the tables with $\lambda = 5$
$P(X < 4) = P(X \leqslant 3) = 0.2650$

b Let n be the number of buses that the company must have to be 99% sure it can fulfil all requests; so $P(X \leqslant n) \geqslant 0.99$.
From the tables with $\lambda = 5$, $P(X \leqslant 10) = 0.9863$, $P(X \leqslant 11) = 0.9945$
So the company needs 11 buses to be 99% sure it can fulfil all requests.

13 a Let X be the number of boats hired in a 30-minute period, so $X \sim Po(4.5)$.
Use the tables with $\lambda = 4.5$
$P(X \geqslant 6) = 1 - P(X \leqslant 5) = 1 - 0.7029 = 0.2971$

b Let Y be the number of boats hired in a 20-minute period, so $Y \sim Po(3)$.
Use the tables with $\lambda = 3$
$P(Y > 8) = 1 - P(Y \leqslant 8) = 1 - 0.9962 = 0.0038$
So the probability that more than 8 boats are requested is 0.38%, which is less than 1%.

c Let n be the number of boats that the company must have to be 99% sure it can meet all demands in a 30-minute period; so $P(X \leqslant n) \geqslant 0.99$.
From the tables with $\lambda = 4.5$, $P(X \leqslant 9) = 0.9829$, $P(X \leqslant 10) = 0.9933$
So the company needs 10 boats to be 99% sure it can fulfil all requests over the hire period.

14 a Let X be the number of breakdowns in a randomly chosen week, so $X \sim Po(1.5)$.
Use the tables with $\lambda = 1.5$
$P(X \leqslant 2) = 0.8088$

b Let Y be the number of breakdowns in a randomly chosen 2-week period, so $Y \sim Po(3)$.
Use the tables with $\lambda = 3$
$P(Y \geqslant 5) = 1 - P(Y \leqslant 4) = 1 - 0.8153 = 0.1847$

14 c Let A be the number of breakdowns in a randomly chosen 6-week period, so $A \sim \text{Po}(9)$.

Let n be the least number of breakdowns so that $\text{P}(X > n) \leqslant 0.05$

$\text{P}(X > n) = 1 - \text{P}(X \leqslant n) \Rightarrow \text{P}(X \leqslant n) = 1 - \text{P}(X > n)$

So find n such that $\text{P}(X \leqslant n) \leqslant 0.95$

From the tables with $\lambda = 4.5$, $\text{P}(X \leqslant 13) = 0.9261$, $\text{P}(X \leqslant 14) = 0.9585$

So the smallest value of n is 14.