S2 UK June 2016 Model Solutions
Kprime2

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1. A student is investigating the numbers of cherries in a *Rays* fruit cake. A random sample of *Rays* fruit cakes is taken and the results are shown in the table below.

Number of cherries	0	1	2	3	4	5	≥ 6
Frequency	24	37	21	12	4	2	0

(a) Calculate the mean and the variance of these data. **(3)**

(b) Explain why the results in part (a) suggest that a Poisson distribution may be a suitable model for the number of cherries in a *Rays* fruit cake. **(1)**

The number of cherries in a *Rays* fruit cake follows a Poisson distribution with mean 1.5

A *Rays* fruit cake is to be selected at random.

Find the probability that it contains

(c) (i) exactly 2 cherries.

(ii) at least 1 cherry. **(4)**

Rays fruit cakes are sold in packets of 5

(d) Show that the probability that there are more than 10 cherries, in total, in a randomly selected packet of *Rays* fruit cakes, is 0.1378 correct to 4 decimal places. **(3)**

Twelve packets of *Rays* fruit cakes are selected at random.

(e) Find the probability that exactly 3 packets contain more than 10 cherries. **(3)**

1(a). $\text{Mean} = \dfrac{37 + 2(21) + 3(12) + 4(4) + 5(2)}{24+37+21+12+4+2} = \cancel{141} \; 1.41$

$\dfrac{\Sigma f x^2}{\Sigma f} - \left(\dfrac{\Sigma f x}{\Sigma f}\right)^2$

$\text{Var} = \dfrac{343}{100} - (1.41)^2 = 1.4419 \approx 1.44$

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Question 1 continued $\mu \approx 1.4$ $var \approx 1.4$

(b) mean $\approx$ Variance ≈ 1.4

$\therefore$ No. of cherries consistently occur at a rate of 1.4 per cake. $\therefore$ Poisson is suited.

(c) Let C = no. of cherries in Ray's fruit cake

$C \sim Po(1.5)$

(i) $P(C=2) = \dfrac{e^{-1.5} \times 1.5^2}{2} = \underline{\underline{0.251 (3sf)}}$

(ii) $P(C \geq 1) = 1 - P(C=0)$

$= 1 - e^{-1.5} = \underline{\underline{0.777 \ (3sf)}}$

(d) Packs of 5 $\Rightarrow$ $1.5 \times 5 = 7.5$ cherries per pack

Let X = no. of total cherries in a selected pack of Rays fruit cake.

$X \sim Po(7.5)$

$P(X > 10) = 1 - P(X \leq 10)$.

$= 1 - 0.8622 = \underline{\underline{0.1378 \ (4dp)}}$

as required.

P 4 6 6 7 4 A 0 3 2 8

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Question 1 continued

(e) $P(X>10) = 0.1378$

Let Y = no. of packets with >10 cherries.

~~$Y \sim Po(0.1378)$~~

$Y \sim Bi(12, 0.1378)$

$P(Y=3) = {}^{12}C_3 \times 0.1378^3 \times 0.8622^9$

$P(Y=3) = 0.152$ (3sf)

4. A continuous random variable X has cumulative distribution function $\mathrm{F}(x)$ given by

$$\mathrm{F}(x) = \begin{cases} 0 & x < 2 \\ k(ax + bx^2 - x^3) & 2 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

Given that the mode of X is $\frac{8}{3}$

(a) show that $b = 8$

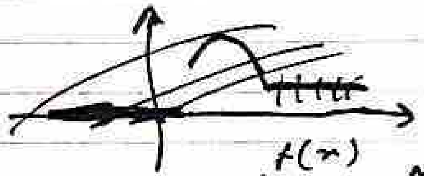
(6)

(b) find the value of k.

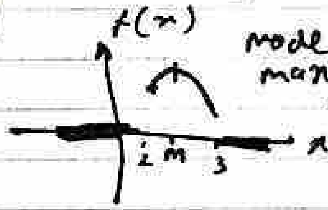
(4)

4(a). $f(x) = \frac{d}{dx}[F(x)] = k(a + 2bx - 3x^2)$

f(x) is a quadratic



Mode is max value of f(x)



$\therefore f'(x) = k(2b - 6x)$

$f'(x) = 0 \Rightarrow k(2b - 6x) = 0$

$\therefore 2b - 6x = 0$

$\therefore x = \frac{b}{3}$

x is the mode

$\therefore \text{mode} = \frac{b}{3} = \frac{8}{3}$

$\Rightarrow b = 8$ as required.

P 4 6 6 7 4 A 0 1 2 2 8

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Question 4 continued

(b) $f(3) = 1$

$\Rightarrow k(3a + 45) = 1$

$f(2) = 0 \Rightarrow k(2a + 24) = 0$

$\therefore 2a + 24 = 0$

$\Rightarrow a = -12$

$a = -12$

$\Rightarrow k = \frac{1}{3a + 45}$

$\therefore k = \frac{1}{9}$

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2. In a region of the UK, 5% of people have red hair. In a random sample of size n, taken from this region, the expected number of people with red hair is 3

(a) Calculate the value of n.

(2)

A random sample of 20 people is taken from this region.
Find the probability that

(b) (i) exactly 4 of these people have red hair,

(ii) at least 4 of these people have red hair.

(5)

Patrick claims that *Reddman* people have a probability greater than 5% of having red hair. In a random sample of 50 *Reddman* people, 4 of them have red hair.

(c) Stating your hypotheses clearly, test Patrick's claim. Use a 1% level of significance.

(5)

2 (a) Let X = no. of people with red hair

$X \sim B(n, 0.05)$

$E(X) = np = 0.05n = 3$

$\Rightarrow n = 60$

(b) $X \sim B(20, 0.05)$

(i) $P(X=4) = {}^{20}C_4 \times 0.05^4 \times 0.95^{16}$

$\therefore P(X=4) = 0.0133$ (3sf)

(ii) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.9841$

$P(X \geq 4) = 0.0159$

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(c) $X \sim B(50, 0.05)$

$H_0: p = 0.05$

$H_1: p > 0.05$

$P(X \geq c) < 0.01$

$\Rightarrow P(X \leq c-1) > 0.99$

$c - 1 = 7$
$\Rightarrow c = 8$

CR is $X \geq 8$

$P(X \geq 4) = 1 - 0.7604 = 0.2396$

$P(X \geq 4) = 0.2396 > 0.01$

$\therefore x = 4$ is not in critical region

$\therefore$ accept H_0

Patrick's Claim is not supported by sufficient evidence. He's incorrect.

Q2

(Total 12 marks)

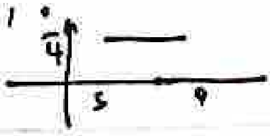
3. The random variable R has a continuous uniform distribution over the interval [5, 9]

(a) Specify fully the probability density function of R. **(1)**

(b) Find $P(7 < R < 10)$ **(1)**

The random variable A is the area of a circle radius R cm.

(c) Find $E(A)$ **(4)**



3(a).

$$f(r) = \begin{cases} 0 & , \text{ otherwise} \\ \frac{1}{4} & , \ 5 \le r \le 9 \end{cases}$$

(b) $P(7 < R < 10) = \frac{2}{4} = \frac{1}{2}$

(c) $A = \pi R^2$

$E(A) = E(\pi R^2) = \pi E(R^2)$

$Var(R) = E(R^2) - [E(R)]^2 \Rightarrow \cancel{\frac{4}{3}}$

$E(R^2) = \frac{4}{3} + 7^2$

$\therefore E(R^2) = \frac{151}{3}$

$\therefore E(A) = \frac{151}{3}\pi$

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5. In a large school, 20% of students own a touch screen laptop. A random sample of n students is chosen from the school. Using a normal approximation, the probability that more than 55 of these n students own a touch screen laptop is 0.0401 correct to 3 significant figures.

Find the value of n.

(8)

5 Let X = no. of students with a touch screen laptop

$X \sim Bi(n, 0.2)$ $np = 0.2n$

$np(1-p) = \cancel{0.2n(}$

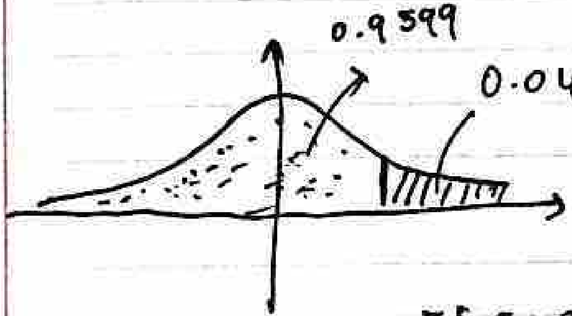
$= 0.16n$

$Y \approx$ # of students with a touch screen laptop.

$Y \sim N(0.2n, 0.16n)$

$P(X > 55) \approx P(Y \geq 55.5)$

$P(Y \geq 55.5) = P\left(Z \geq \dfrac{55.5 - 0.2n}{\frac{2}{5}\sqrt{n}}\right) = 0.0401$



$\Rightarrow P\left(Z \leq \dfrac{55.5 - 0.2n}{\frac{2}{5}\sqrt{n}}\right) = 0.9599$

$\therefore \dfrac{55.5 - 0.2n}{\frac{2}{5}\sqrt{n}} = 1.75$ (from tables)

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Question 5 continued

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$\therefore 55.5 - 0.2n = 0.7\sqrt{n}$

$(\times 10) \Rightarrow 555 - 2n = 7\sqrt{n}$

$\therefore 2n + 7\sqrt{n} - 555 = 0$

$\therefore \sqrt{n} = \dfrac{-b \pm b^2 - 4ac}{2a}$

$(\sqrt{n} - 15)(2\sqrt{n} + 37) = 0$

$\sqrt{n} = 15$ or $\sqrt{n} = -\dfrac{37}{2}$

$n = 225$ or $n = 342.25$

$\Rightarrow n = 225$

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6. A bag contains a large number of counters with one of the numbers 4, 6 or 8 written on each of them in the ratio 5 : 3 : 2 respectively.

A random sample of 2 counters is taken from the bag.

(a) List all the possible samples of size 2 that can be taken.
(2)

The random variable M represents the mean value of the 2 counters.

Given that $P(M = 4) = \frac{1}{4}$ and $P(M = 8) = \frac{1}{25}$

(b) find the sampling distribution for M.
(5)

A sample of n sets of 2 counters is taken. The random variable Y represents the number of these n sets that have a mean of 8

(c) Calculate the minimum value of n such that $P(Y \geqslant 1) > 0.9$
(3)

6(a) (4, 4) (4, 6) (6, 4)
(6, 6) (4, 8) (8, 4)
(8, 8) (6, 8) (8, 6)

~~(b) 4 6 8~~
~~5/9 1/3 2/9~~

(b)

Counter	4	6	8
Probability	1/2	3/10	1/5

(4, 4)

$P(M = 4) = \frac{1}{4}$ $P(M = 8) = \frac{1}{25}$

P 4 6 6 7 4 A 0 2 0 2 8

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Question 6 continued

$(4,4) \quad | \quad P(M=4) = \frac{1}{4}$

~~$(6,6) \quad P(M=3) = \left(\frac{3}{10}\right)^2 = \frac{9}{100}$~~

$\left.\begin{array}{l}(6,6)\\(4,8)\\(8,4)\end{array}\right\} \quad P(M=6) = \left(\frac{3}{10}\right)^2 + 2\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)$

$= \frac{29}{100}$

$\left.\begin{array}{l}(4,6)\\(6,4)\end{array}\right\} \quad P(M=5) = 2 \times \frac{1}{2} \times \frac{3}{10} = \frac{3}{10}$

$\left.\begin{array}{l}(6,8)\\(8,6)\end{array}\right\} \quad P(M=7) = 2 \times \frac{3}{10} \times \frac{1}{5} = \frac{3}{25}$

M	4	5	6	7	8
$P(M=m)$	$1/4$	$\frac{3}{10}$	$\frac{29}{100}$	$\frac{3}{25}$	$\frac{1}{25}$

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(c) $P(\text{1 set of two counters has } M=8) = \frac{1}{25}$

$Y \sim Bi\left(n, \frac{1}{25}\right)$

$P(Y \geq 1) = 1 - P(Y=0) = 1 - \left(\frac{24}{25}\right)^n$

$= \cancel{1}$

$\therefore \quad 1 - P(Y=0) = 1 - \left(\frac{24}{25}\right)^n > 0.9$

$\therefore \quad \left(\frac{24}{25}\right)^n < 0.1$

$\therefore \quad n \ln\left(\frac{24}{25}\right) < \ln 0.1$

$\therefore \quad n > 56.405\ldots$

$\therefore \quad n = 57$

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7. The weight, X kg, of staples in a bin full of paper has probability density function

$$f(x) = \begin{cases} \dfrac{9x - 3x^2}{10} & 0 \leqslant x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Use integration to find

(a) $E(X)$ **(4)**

(b) $\text{Var}(X)$ **(4)**

(c) $P(X > 1.5)$ **(3)**

Peter raises money by collecting paper and selling it for recycling. A bin full of paper is sold for £50 but if the weight of the staples exceeds 1.5 kg it sells for £25

(d) Find the expected amount of money Peter raises per bin full of paper. **(2)**

Peter could remove all the staples before the paper is sold but the time taken to remove the staples means that Peter will have 20% fewer bins full of paper to sell.

(e) Decide whether or not Peter should remove all the staples before selling the bins full of paper. Give a reason for your answer. **(2)**

7(a) $E(X) = \int_0^2 x\, f(x)\, dx$

$= \frac{1}{10} \int_0^2 9x^2 - 3x^3\, dx$

$= \frac{1}{10} \left[3x^3 - \frac{3}{4} x^4 \right]_0^2$

$\therefore E(X) = \frac{6}{5}$

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Question 7 continued

(b) $E(X^2) = \int_0^2 x^2 f(x)\, dx$

$= \frac{1}{10}\int_0^2 9x^3 - 3x^4 \, dx$

$= \frac{1}{10}\left[\frac{9}{4}x^4 - \frac{3}{5}x^5\right]_0^2$

$= \frac{42}{25}$

$\therefore Var(X) = \frac{42}{25} - \left(\frac{6}{5}\right)^2 = \frac{6}{25}$

(c) $P(X > 1.5) = \int_{1.5}^2 \frac{9x - 3x^2}{10}\, dx$

$= \frac{1}{10}\left[\frac{9}{2}x^2 - x^3\right]_{1.5}^2$

$= \frac{13}{40}$

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Question 7 continued

(d) $+£50$ / bin full of paper

$+£25$ / bin if weight exceeds 1.5kg

$P(X<1.5) = \frac{27}{40}$ $P(X>1.5) = \frac{13}{40}$

$\therefore E\binom{\text{money}}{\text{raised}} = 50 \times \frac{27}{40} + 25 \times \frac{13}{40}$

$= £41.88$

(e) $E\begin{pmatrix}\text{money raised} \\ \text{if Peter removes} \\ \text{all staples per} \\ \text{bin}\end{pmatrix} = 50 \times 0.8 = £40$

$E\begin{pmatrix}\text{money raised} \\ \text{if Peter keeps} \\ \text{the staples}\end{pmatrix} = £41.88$ from part (d)

$\therefore$ Peter should not remove all the staples before selling the bins full of paper, since profit $£41.88 > £40$ per bin

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