S2 S13 R

1. A bag contains a large number of counters. A third of the counters have a number 5 on them and the remainder have a number 1.

A random sample of 3 counters is selected.

(a) List all possible samples.

(2)

(b) Find the sampling distribution for the range.

(3)

a) 5,5,5

5,5,1 5,1,5 1,5,5

5,1,1 1,5,1 1,1,5

1,1,1

range = 0 $P(5,5,5) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

range = 4 $P = 3 \times \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{2}{9}$

range = 4 $P = 3 \times \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4}{9}$

range = 0 $P = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

b)

Range	0	4
P	$\frac{2}{3}$	$\frac{1}{3}$

2. The continuous random variable Y has cumulative distribution function

$$\mathrm{F}(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{4}(y^3 - 4y^2 + ky) & 0 \leqslant y \leqslant 2 \\ 1 & y > 2 \end{cases}$$

where k is a constant.

(a) Find the value of k.

(2)

(b) Find the probability density function of Y, specifying it for all values of y.

(3)

(c) Find $\mathrm{P}(Y > 1)$.

(2)

a) $F(0) = 0 \Rightarrow 0 = 0$ $\quad F(2) = 1 \Rightarrow \frac{1}{4}(8 - 16 + 2k) = 1$

$-8 + 2k = 4 \quad 2k = 12$

$\underline{k = 6}$

b) $f(y) = \frac{d}{dx}F(y) = \frac{1}{4}(3y^2 - 8y + 6)$

$\therefore \; f(y) = \begin{cases} \frac{1}{4}(3y^2 - 8y + 6) & 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$

c) $P(y > 1) \quad P(y \geq 2) = 1 - P(y \leq 1) = 1 - F(1)$

$= 1 - \frac{1}{4}(1 - 4 + 6) = 1 - \frac{3}{4} = \underline{\frac{1}{4}}$

3. The random variable X has a continuous uniform distribution on $[a, b]$ where a and b are positive numbers.

Given that $E(X) = 23$ and $\text{Var}(X) = 75$

(a) find the value of a and the value of b.

(6)

Given that $P(X > c) = 0.32$

(b) find $P(23 < X < c)$.

(2)

$E(x) = \frac{a+b}{2} = 23 \Rightarrow a+b=46 \Rightarrow b = 46-a$

$V(x) = \frac{(b-a)^2}{12} = 75 \Rightarrow (b-a)^2 = 900 \Rightarrow b-a=30$

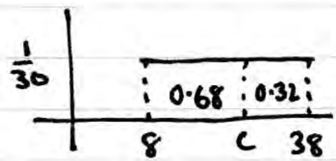
$b+a=46$

$2a = 16$

$a = 8$

$b = 38$

b)



$\therefore P(23 < X < 28.4)$

$= \frac{5.4}{30} = 0.18$

$\frac{c-8}{30} = 0.68$

$\Rightarrow c-8 = 20.4$

$\therefore c = 28.4$

blank

4. The random variable X has probability density function f(x) given by

$$f(x) = \begin{cases} k(3 + 2x - x^2) & 0 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that $k = \dfrac{1}{9}$ **(3)**

(b) Find the mode of X. **(2)**

(c) Use algebraic integration to find E(X). **(4)**

By comparing your answers to parts (b) and (c),

(d) describe the skewness of X, giving a reason for your answer. **(2)**

a) $\int f(x)dx = 1 \Rightarrow k\,[3x + x^2 - \frac{1}{3}x^3]_0^3 = 9k = 1 \quad \therefore k = \frac{1}{9}$ 2

b) mode when $f'(x) = 0 \quad f'(x) = \frac{1}{9}(2 - 2x) = 0$

$\therefore x = 1$ 2

c) $E(X) = \int_0^3 x f(x)\,dx = \frac{1}{9}\int_0^3 3x + 2x^2 - x^3\,dx = \frac{1}{9}[\frac{3x^2}{2} + \frac{2}{3}x^3 - \frac{1}{4}x^4]_0^3$

$E(X) = \frac{5}{4}$

d) mean $(\frac{5}{4})$ > mode (1) ∴ positive skew 2

5. In a village shop the customers must join a queue to pay. The number of customers joining the queue in a 10 minute interval is modelled by a Poisson distribution with mean 3

Find the probability that

(a) exactly 4 customers join the queue in the next 10 minutes,

(2)

(b) more than 10 customers join the queue in the next 20 minutes.

(3)

When a customer reaches the front of the queue the customer pays the assistant. The time each customer takes paying the assistant, T minutes, has a continuous uniform distribution over the interval [0, 5]. The random variable T is independent of the number of people joining the queue.

(c) Find $P(T > 3.5)$

(1)

In a random sample of 5 customers, the random variable C represents the number of customers who took more than 3.5 minutes paying the assistant.

(d) Find $P(C \geqslant 3)$

(3)

Bethan has just reached the front of the queue and starts paying the assistant.

(e) Find the probability that in the next 4 minutes Bethan finishes paying the assistant and no other customers join the queue.

(4)

a) $X \sim Po(3)$ $\quad P(X=4) = \frac{e^{-3} \times 3^4}{4!} = 0.168$

b) $P(Y>10)$ $\quad Y \sim Po(6)$ $\quad 1-P(Y \leq 10) = 1-0.9574$
$P(Y \geq 11)$ $\quad = 0.0426$

c) $T \sim U[0,5]$ $\quad P(T>3.5) = \frac{1.5}{5} = 0.3$

d) $C \sim B(5, 0.3)$ $\quad P(C \geq 3)$ $P(C>2) = 1-P(C \leq 2)$
$= 1 - 0.8369 = 0.1631$

e) $P(C \leq 4) = \frac{4}{5}$ $\quad X \sim Po(\frac{3}{10} \times 4)$ $X \sim Po(1.2)$ $\quad$ x now = customers in 4min

$P(X=0) = e^{-1.2} = 0.301$

$P(C \leq 4) \cap P(X=0) = \frac{4}{5} \times 0.301 = 0.241$

6. Frugal bakery claims that their packs of 10 muffins contain on average 80 raisins per pack. A Poisson distribution is used to describe the number of raisins per muffin.

A muffin is selected at random to test whether or not the mean number of raisins per muffin has changed.

(a) Find the critical region for a two-tailed test using a 10% level of significance. The probability of rejection in each tail should be less than 0.05

(4)

(b) Find the actual significance level of this test.

(2)

The bakery has a special promotion claiming that their muffins now contain even more raisins.

A random sample of 10 muffins is selected and is found to contain a total of 95 raisins.

(c) Use a suitable approximation to test the bakery's claim. You should state your hypotheses clearly and use a 5% level of significance.

(8)

a) x = raisins per muffin $x \sim Po(8)$

$P(x \leq L) < 0.05$ $P(x \geq u) < 0.05$ $P(x > u-1)$

$P(x \leq 3) = 0.0424$ $1 - P(x \leq u-1) < 0.05$

$P(x \leq 4) = 0.0996$ $P(x \leq u-1) > 0.95$

$P(x \leq 12) = 0.936$ $P(x \leq 13) = 0.9658$

$\therefore L = 3$ $\therefore u-1 = 13$ $\therefore u = 14$

CR $x \leq 3 \cup x \geq 14$

b) $ASL = 0.0424 + 0.0342 = 0.0766$ (7.66%)

c) y = raisins per pack of 10 muffins $y \sim Po(80)$ $\mu = 80$ $\sigma^2 = 80$

$H_0 : \lambda = 80$ $\approx y \sim N(80, 80)$ $P(y \geq 95)$

$H_1 : \lambda > 80$ $P(y > 94)$ cc $P(y > 94.5)$

$\approx P(Z > \frac{94.5 - 80}{\sqrt{80}}) \approx P(Z > 1.62) = 1 - \Phi(1.62) = 0.0526$

not < 0.05 $\therefore$ not enough evidence to reject null hypothesis as test is not statistically significant

$\therefore$ not enough evidence to support claim

7. As part of a selection procedure for a company, applicants have to answer all 20 questions of a multiple choice test. If an applicant chooses answers at random the probability of choosing a correct answer is 0.2 and the number of correct answers is represented by the random variable X.

(a) Suggest a suitable distribution for X.

(2)

Each applicant gains 4 points for each correct answer but loses 1 point for each incorrect answer. The random variable S represents the final score, in points, for an applicant who chooses answers to this test at random.

(b) Show that $S = 5X - 20$

(2)

(c) Find E(S) and Var(S).

(4)

An applicant who achieves a score of at least 20 points is invited to take part in the final stage of the selection process.

(d) Find P($S \geqslant 20$)

(4)

Cameron is taking the final stage of the selection process which is a multiple choice test consisting of 100 questions. He has been preparing for this test and believes that his chance of answering each question correctly is 0.4

(e) Using a suitable approximation, estimate the probability that Cameron answers more than half of the questions correctly.

(5)

a) $X \sim B(20, 0.2)$ Binomial

b) Mean correct $= 20 \times 0.8 = 16$ $\quad 16 \times 4 = 64$ points

$\Rightarrow 4$ wrong $\quad -4 \Rightarrow 60$ points

$x =$ Correct $\therefore 20 - x =$ Wrong $\quad S = 4x - (20 - x) = 5x - 20$ #

a) $X \sim B(20, 0.2)$ Binomial

b) $E(x) = np = 4 \quad V(x) = np(1-p) = 4(0.8) = 0.32$

$x = \text{correct} \therefore 20 - x = \text{wrong} \Rightarrow S = x \times 4 - 1(20 - x) = 5x - 20$

2

c) $E(S) = 5(E(x)) - 20 = 0$

$V(S) = 5^2 V(x) - \cancel{20} = 25 \times 0.32 = 8$

d) $S \geq 20 \Rightarrow 5x - 20 \geq 20 \Rightarrow 5x \geq 40 \quad x \geq 8$

$P(x \geq 8)$
$P(x > 7)$ $= 1 - P(x \leq 7) = 1 - 0.9679 = 0.0321$

2

e) y = correct answer in final stage

$y \sim B(100, 0.4) \quad \mu = 40 \quad \sigma^2 = 40(1 - 0.4) = 24$

$\approx N(40, 24) \quad P(y > 50) \quad P(y \geq 51) \quad cc \quad P(y > 50.5)$

$\approx P\left(Z > \frac{50.5 - 40}{\sqrt{24}}\right) \approx P(Z > 2.14) = 1 - \Phi(2.14) = 0.0162$

2