S2 W11

1. A disease occurs in 3% of a population.

(a) State any assumptions that are required to model the number of people with the disease in a random sample of size n as a binomial distribution.
(2)

(b) Using this model, find the probability of exactly 2 people having the disease in a random sample of 10 people.
(3)

(c) Find the mean and variance of the number of people with the disease in a random sample of 100 people.
(2)

A doctor tests a random sample of 100 patients for the disease. He decides to offer all patients a vaccination to protect them from the disease if more than 5 of the sample have the disease.

(d) Using a suitable approximation, find the probability that the doctor will offer all patients a vaccination.
(3)

a) Probability of having the disease is constant
the probability of having the disease is independent.

b) $x \sim B(10, 0.03)$ $\quad P(X=2) = \binom{10}{2} 0.03^2 0.97^8$
$= 0.0317$

c) Mean $= np = 100 \times 0.03 = 3$

Variance $= np(1-p) = 3(0.97) = 2.91$

d) $np = 3 \therefore \approx y \sim Po(3)$

$P(y > 5) = 1 - P(y \leq 5) = 0.0839$

2. A student takes a multiple choice test. The test is made up of 10 questions each with 5 possible answers. The student gets 4 questions correct. Her teacher claims she was guessing the answers. Using a one tailed test, at the 5% level of significance, test whether or not there is evidence to reject the teacher's claim.
State your hypotheses clearly.
(6)

$x \sim B(10, 0.2)$ $\quad \begin{matrix} P(x \geq 4) \\ P(x > 3) \end{matrix} \Rightarrow 1 - P(x \leq 3)$

$H_0 : P = 0.2$ $\quad = 0.1209$

$H_1 : P > 0.2$ $\quad (> 0.05)$

not enough evidence to reject null hypothesis as test is not significant
$\therefore$ not enough evidence to reject teacher's claim.

3. The continuous random variable X is uniformly distributed over the interval $[-1,3]$.
Find

(a) $E(X)$
(1)

(b) $Var(X)$
(2)

(c) $E(X^2)$
(2)

(d) $P(X < 1.4)$
(1)

A total of 40 observations of X are made.

(e) Find the probability that at least 10 of these observations are negative.
(5)

a) $E(x) = \frac{a+b}{2} = 1$ $\quad$ b) $V(x) = \frac{(b-a)^2}{12} = \frac{16}{12} = \frac{4}{3}$

c) $V(x) = E(x^2) - E(x)^2 \Rightarrow E(x^2) = \frac{4}{3} + 1^2 = \frac{7}{3}$

d) $P(x < 1.4) = \frac{2.4}{4} = 0.6$

e) $P(x < 0) = \frac{1}{4} \Rightarrow y \sim B(40, \frac{1}{4})$

$\begin{matrix} P(y \geq 10) \\ P(y > 9) \end{matrix} = 1 - P(y \leq 9) = 0.5605$

4. Richard regularly travels to work on a ferry. Over a long period of time, Richard has found that the ferry is late on average 2 times every week. The company buys a new ferry to improve the service. In the 4-week period after the new ferry is launched, Richard finds the ferry is late 3 times and claims the service has improved. Assuming that the number of times the ferry is late has a Poisson distribution, test Richard's claim at the 5% level of significance. State your hypotheses clearly.

(6)

$x \sim Po(8)$ x = lates in 4-week period

$H_0 : \lambda = 8$ $P(x \leq 3) = 0.0424$ (< 0.05)

$H_1 : \lambda < 8$

$\therefore$ there is enough evidence to reject null hypothesis since result is significant

$\therefore$ enough evidence, to support claim

5. A continuous random variable X has the probability density function f(x) shown in Figure 1.

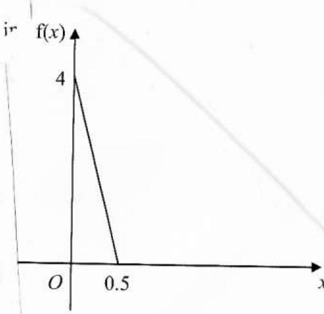

Figure 1

(a) Show that $f(x) = 4 - 8x$ for $0 \leqslant x \leqslant 0.5$ and specify f(x) for all real values of x. **(4)**

(b) Find the cumulative distribution function F(x). **(4)**

(c) Find the median of X. **(3)**

(d) Write down the mode of X. **(1)**

(e) State, with a reason, the skewness of X. **(1)**

a) $m = \frac{-4}{0.5} = -8$ $(0, 4)$

$\therefore y - 4 = -8x \Rightarrow f(x) = -8x + 4$

$f(x) = \begin{cases} -8x + 4 & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$

b) $F(x) = \int f(x)\,dx = \int 4 - 8x\,dx = 4x - 4x^2 + c$

$x = 0$ $F(0) = 0$ $\therefore c = 0$

$F(x) = \begin{cases} 0 & x < 0 \\ 4x(1-x) & 0 \leq x \leq 0.5 \\ 1 & x > 0.5 \end{cases}$

c) $F(Q_2) = 0.5 \Rightarrow 4x - 4x^2 = 0.5$

$\Rightarrow 8x^2 - 8x + 1 = 0$ $\therefore x = Q_2 = \underline{0.146}$

d) mode $= 0$

e) median > mode $\therefore$ Positive skew.

6. Cars arrive at a motorway toll booth at an average rate of 150 per hour.

(a) Suggest a suitable distribution to model the number of cars arriving at the toll booth, X, per minute. **(2)**

(b) State clearly any assumptions you have made by suggesting this model. **(2)**

Using your model,

(c) find the probability that in any given minute

(i) no cars arrive,

(ii) more than 3 cars arrive. **(3)**

(d) In any given 4 minute period, find m such that $P(X > m) = 0.0487$ **(3)**

(e) Using a suitable approximation find the probability that fewer than 15 cars arrive in any given 10 minute period. **(6)**

a) x = Cars per min $x \sim Po(2.5)$

b) cars arrive one at a time at a constant rate, cars arriving are Independent of one another

c) $P(x = 0) = e^{-2.5} = 0.082$

ii) $P(x > 3) = 1 - P(x \leq 3) = 0.2424$

d) $y \sim Po(10)$ $P(y > m) = 0.0487$

$\Rightarrow 1 - P(y \leq m) = 0.0487$ $\therefore P(y \leq m) = 0.9513$

$\therefore \underline{m = 15}$

e) $C \sim Po(25) \approx C \sim N(25, 25)$

$P(C < 15)$, $P(C \leq 14)$ cc $P(C < 14.5) \approx P\left(Z < \frac{14.5 - 25}{\sqrt{25}}\right)$

$\approx P(Z < -2.1) = 1 - \Phi(2.1) = 0.0176$

7. The queuing time in minutes, X, of a customer at a post office is modelled by the probability density function

$$f(x) = \begin{cases} kx(81 - x^2) & 0 \leqslant x \leqslant 9 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $k = \dfrac{4}{6561}$.

(3)

Using integration, find

(b) the mean queuing time of a customer,

(4)

(c) the probability that a customer will queue for more than 5 minutes.

(3)

Three independent customers shop at the post office.

(d) Find the probability that at least 2 of the customers queue for more than 5 minutes.

(3)

a) $\int f(x)\,dx = 1 \quad k\int_0^9 81x - x^3\,dx = 1$

$\Rightarrow k\left[\frac{81}{2}x^2 - \frac{1}{4}x^4\right]_0^9 = 1 \Rightarrow \frac{6561k}{4} = 1 \quad \therefore k = \frac{4}{6561}$

b) $E(X) = \int x f(x)\,dx = \frac{4}{6561}\int_0^9 81x^2 - x^4\,dx$

$= \frac{4}{6561}\left[27x^3 - \frac{1}{5}x^5\right]_0^9 = 4.8$

blank

Question 7 continued

c) $P(X > 5) = \int_5^9 \frac{4}{6561}(81x - x^3)\,dx$

$= \frac{4}{6561}\left[\frac{81}{2}x^2 - \frac{1}{4}x^4\right]_5^9 = \frac{4}{6561}\left[\frac{6561}{4} - \frac{3425}{4}\right]$

$\Rightarrow P(X > 5) = 0.478$

d)

Taking the probability that at least 2 queue for more than 5 minutes =

$3(1 - 0.478)(0.478)^2 + (0.478)^3$

$= 0.467$