S2 W10

1. A manufacturer supplies DVD players to retailers in batches of 20. It has 5% of the players returned because they are faulty.

(a) Write down a suitable model for the distribution of the number of faulty DVD players in a batch. **(2)**

Find the probability that a batch contains

(b) no faulty DVD players, **(2)**

(c) more than 4 faulty DVD players. **(2)**

(d) Find the mean and variance of the number of faulty DVD players in a batch. **(2)**

a) x = faulty DVD Player $x \sim B(20, 0.05)$

b) $P(x=0) = 0.95^{20} = \underline{0.358}$

c) $P(x>4) = 1 - P(x \leq 4) = \underline{0.0026}$

d) Mean $= np = 20 \times 0.05 = 1$
Variance $= np(1-p) = 1(0.95) = \underline{0.95}$

2. A continuous random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0, & x < -2 \\ \frac{x+2}{6}, & -2 \leqslant x \leqslant 4 \\ 1, & x > 4 \end{cases}$$

(a) Find $P(X<0)$. **(2)**

(b) Find the probability density function $f(x)$ of X. **(3)**

(c) Write down the name of the distribution of X. **(1)**

(d) Find the mean and the variance of X. **(3)**

(e) Write down the value of $P(X = 1)$. **(1)**

a) $P(x<0) = F(0) = \frac{2}{6} = \frac{1}{3}$

b) $f(x) = \frac{d}{dx}F(x) = \frac{1}{6}$ $\quad f(x) = \begin{cases} \frac{1}{6} & -2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$

c) Continuous uniform distribution

$x \sim U[-2, 4]$

d) $E(X) = 1$ $\quad V(X) = \frac{(4+2)^2}{12} = 3$ $\quad$ e) $P(x=1) = 0$

3. A robot is programmed to build cars on a production line. The robot breaks down at random at a rate of once every 20 hours.

(a) Find the probability that it will work continuously for 5 hours without a breakdown. **(3)**

Find the probability that, in an 8 hour period,

(b) the robot will break down at least once, **(3)**

(c) there are exactly 2 breakdowns. **(2)**

In a particular 8 hour period, the robot broke down twice.

(d) Write down the probability that the robot will break down in the following 8 hour period. Give a reason for your answer. **(2)**

a) x = robot breaks down $x \sim Po(0.25)$

$P(x=0) = e^{-0.25} = \underline{0.7788}$

b) $x \sim Po(0.4)$ $\quad P(x \geq 1) = 1 - P(x=0)$

$= 1 - e^{-0.4} = \underline{0.3297}$

c) $P(x=2) = \frac{e^{-0.4} \times 0.4^2}{2} = \underline{0.0536}$

d) 0.3297 Since Poisson events are independent.

4. The continuous random variable X has probability density function f(x) given by

$$f(x) = \begin{cases} k(x^2 - 2x + 2) & 0 < x \leqslant 3 \\ 3k & 3 < x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that $k = \frac{1}{9}$ **(4)**

(b) Find the cumulative distribution function F(x). **(6)**

(c) Find the mean of X. **(3)**

(d) Show that the median of X lies between $x = 2.6$ and $x = 2.7$ **(4)**

a) $\int f(x)dx = 1 \therefore k\int_0^3 x^2-2x+2\,dx + k\int_3^4 3\,dx = 1$

$k\left[\frac{1}{3}x^3 - x^2 + 2x\right]_0^3 + k\left[3x\right]_3^4 = 1$

$k\left[(9-9+6)-(0)+(12)-(9)\right] = 1 \quad \therefore 9k = 1 \therefore k = \frac{1}{9}$

b) $x \leq 3 \quad F(x) = \frac{1}{9}\int_0^x t^2 - 2t + 2\,dt = \frac{1}{9}\left[\frac{1}{3}t^3 - t^2 + 2t\right]_0^x$

$= \frac{1}{27}x^3 - \frac{1}{9}x^2 + \frac{2}{9}x \qquad F(3) = 1 - 1 + \frac{6}{9} = \frac{2}{3}$

$x \leq 4 \quad F(x) = \int_3^x \frac{1}{3}dt + \frac{2}{3} = \left[\frac{1}{3}t\right]_3^x + \frac{2}{3} = \frac{1}{3}x - \frac{1}{3}$

$$\therefore F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{27}x^3 - \frac{1}{9}x^2 + \frac{2}{9}x & 0 < x \leq 3 \\ \frac{1}{3}x - \frac{1}{3} & 3 < x \leq 4 \\ 1 & x > 4 \end{cases}$$

c) $E(X) = \int x f(x)dx = \frac{1}{9}\int_0^3 x^3 - 2x^2 + 2x\,dx + \frac{1}{3}\int_3^4 x\,dx$

$= \frac{1}{9}\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + x^2\right]_0^3 + \frac{1}{3}\left[\frac{1}{2}x^2\right]_3^4$

$= \frac{1}{9}\left(\frac{45}{4}\right) + \frac{1}{3}\left(\frac{7}{2}\right) = \frac{5}{4} + \frac{7}{6} = 2.42$

Question 4 continued

$F(2.6) = 0.478 < 0.5 \qquad F(Q_2) = 0.5$

$F(2.7) = 0.519 > 0.5 \qquad \therefore 2.6 < Q_2 < 2.7$

5. A café serves breakfast every morning. Customers arrive for breakfast at random at a rate of 1 every 6 minutes.

Find the probability that

(a) fewer than 9 customers arrive for breakfast on a Monday morning between 10 am and 11 am. **(3)**

The café serves breakfast every day between 8 am and 12 noon.

(b) Using a suitable approximation, estimate the probability that more than 50 customers arrive for breakfast next Tuesday. **(6)**

a) $X \sim Po(10) \quad P(X < 9) = P(X \leq 8) = 0.3328$

b) $X \sim Po(40) \approx N(40, 40)$

$P(X > 50)$
$P(X \geq 51)$ cc $P(X > 50.5) \approx P\left(Z > \frac{50.5 - 40}{\sqrt{40}}\right)$

$P(Z > 1.66) = 1 - \Phi(1.66) = 0.0485$

6. (a) Define the critical region of a test statistic. **(2)**

A discrete random variable X has a Binomial distribution $B(30, p)$. A single observation is used to test $H_0 : p = 0.3$ against $H_1 : p \neq 0.3$

(b) Using a 1% level of significance find the critical region of this test. You should state the probability of rejection in each tail which should be as close as possible to 0.005 **(5)**

(c) Write down the actual significance level of the test. **(1)**

The value of the observation was found to be 15.

(d) Comment on this finding in light of your critical region. **(2)**

a) range of values of the test statistic for which there is enough evidence to reject the null hypothesis

b) $H_0 : p = 0.3$ $\quad x \sim B(30, 0.3)$
$H_1 : p \neq 0.3$

$P(x \leq L) \approx 0.005$
$P(x \leq 2) = 0.0021$
$P(x \leq 3) = 0.0093$
$\therefore L = 2$

$P(x \geq U) \approx 0.005$
$P(x > U-1) \approx 0.005$
$1 - P(x \leq U-1) \approx 0.005$
$P(x \leq U-1) \approx 0.995$
$P(x \leq 15) \approx 0.9936$
$P(x \leq 16) = 0.9979$
$\therefore U-1 = 15 \quad \therefore U = 16$

$\therefore$ CR $\{x \leq 2\} \cup \{x \geq 16\}$

c) ASL $= 0.0021 + 0.0064 = 0.0085 \quad \therefore 8.5\%$

d) 15 does not lie in the critical region
$\therefore$ result is not significant
$\therefore$ not enough evidence to reject null hypothesis
$\therefore$ not enough evidence to suggest $p \neq 0.3$

7. A bag contains a large number of coins. It contains only 1p and 2p coins in the ratio 1:3

(a) Find the mean μ and the variance σ^2 of the values of this population of coins. **(3)**

A random sample of size 3 is taken from the bag.

(b) List all the possible samples. **(2)**

(c) Find the sampling distribution of the mean value of the samples. **(6)**

a)

x	1	2
p	0.25	0.75

$E(x) = 0.25 + 1.5 = 1.75$

$E(x^2) = 0.25 + 3 = 3.25$

$\therefore V(x) = 3.25 - 1.75^2 = 0.1875$

b) 1, 1, 1 $\quad \mu = 1 \quad p = 0.25^3$

1, 1, 2 $\quad$ 1, 2, 1 $\quad$ 2, 1, 1 $\quad \mu = \frac{4}{3} \quad p = 3 \times 0.25^2 \times 0.75$

1, 2, 2 $\quad$ 2, 1, 2 $\quad$ 2, 2, 1 $\quad \mu = \frac{5}{3} \quad p = 3 \times 0.25 \times 0.75^2$

2, 2, 2 $\quad \mu = 2 \quad p = 0.75^3$

c)

μ	1	$\frac{4}{3}$	$\frac{5}{3}$	2
p	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{27}{64}$	$\frac{27}{64}$