Exercise 7D

1 $X \sim \mathrm{N}\left(20, 4^2\right)$

a $\mathrm{P}(X \leqslant 26)$

$$z = \frac{X-\mu}{\sigma}$$
$$= \frac{26-20}{4}$$
$$= 1.5$$

$\mathrm{P}(X \leqslant 26) = \mathrm{P}(Z \leqslant 1.5)$

$\Phi(1.5) = 0.9332$

$\mathrm{P}(X \leqslant 26) = 0.9332$

b $\mathrm{P}(X > 30) = 1 - \mathrm{P}(X \leqslant 30)$

$$z = \frac{X-\mu}{\sigma}$$
$$= \frac{30-20}{4}$$
$$= 2.5$$

$\mathrm{P}(X > 30) = 1 - \mathrm{P}(Z \leqslant 2.5)$

$\Phi(2.5) = 0.9938$

$\mathrm{P}(X > 30) = 1 - 0.9938$

$= 0.0062$

c $\mathrm{P}(X \geq 17) = \mathrm{P}(X \leqslant 23)$

$$z = \frac{X-\mu}{\sigma}$$
$$= \frac{23-20}{4}$$
$$= 0.75$$

$\Phi(0.75) = 0.7734$

$\mathrm{P}(X \geq 17) = 0.7734$

2 $X \sim \mathrm{N}(18, 10)$

a $\mathrm{P}(X > 20) = 1 - \mathrm{P}(X \leqslant 20)$

$$z = \frac{X-\mu}{\sigma}$$
$$= \frac{20-18}{\sqrt{10}}$$
$$= 0.6325$$

$\Phi(0.6325) = 0.7365$

$\mathrm{P}(X > 20) = 1 - 0.7365$

$= 0.2635$

2 b $P(X < 15) = 1 - P(X < 21)$

$z = \frac{X - \mu}{\sigma}$

$= \frac{21 - 18}{\sqrt{10}}$

$= 0.9487$

$\Phi(0.9487) = 0.8286$

$P(X < 15) = 1 - 0.8286$

$= 0.1714$

3 a $X \sim N(24, 3^2)$

$P(X \leqslant 29)$

$z = \frac{X - \mu}{\sigma}$

$= \frac{29 - 24}{3}$

$= 1.667$

$\Phi(1.667) = 0.9522$

$P(X \leqslant 29) = 0.9522$

b $P(X \geq 22) = P(X \leqslant 26)$

$z = \frac{X - \mu}{\sigma}$

$= \frac{26 - 24}{3}$

$= 0.6667$

$\Phi(0.6667) = 0.7475$

$P(X \geq 22) = 0.7475$

c $P(X < 16) = P(X > 32) = 1 - P(X \leqslant 32)$

$z = \frac{X - \mu}{\sigma}$

$= \frac{32 - 24}{3}$

$= 2.667$

$\Phi(2.667) = 0.9962$

$P(X < 16) = 1 - 0.9962$

$= 0.0038$

4 $Y \sim \mathrm{N}(30, 5^2)$

$\mathrm{P}\left(Z > \frac{a-30}{5}\right) = 0.30 \Rightarrow \mathrm{P}\left(Z < \frac{a-30}{5}\right) = 0.70$

$\frac{a-30}{5} = 0.5244$

$a = 32.622$

5 $Y \sim \mathrm{N}(15, 3^2)$

$\mathrm{P}\left(z > \frac{a-15}{3}\right) = 0.15 \Rightarrow \mathrm{P}\left(z < \frac{a-15}{3}\right) = 0.85$

$\frac{a-15}{3} = 1.036$

$a = 18.108$

$= 18.1$ (to 3 s.f.)

6 $Y \sim \mathrm{N}(100, 15^2)$

a $\mathrm{P}(Y > s) = 0.975$

$\mathrm{P}\left(z > \frac{s-100}{15}\right) = 0.975 \Rightarrow \mathrm{P}\left(z < \frac{m-100}{15}\right) = 0.975$

where m lies the same distance to the right of the mean as s does to the left

$\frac{m-100}{15} = 1.960$

$m = 129.4$

Therefore,

$s = 100 - 29.4$

$= 70.6$

b $\mathrm{P}(Y < t) = 0.10$

$\mathrm{P}\left(z < \frac{t-100}{15}\right) = 0.10 \Rightarrow \mathrm{P}\left(z > \frac{n-100}{15}\right) = 0.1 \Rightarrow \mathrm{P}\left(z < \frac{n-100}{15}\right) = 0.9$

where n lies the same distance to the right of the mean as t does to the left

$\frac{n-100}{15} = 1.282$

$n = 119.23$

Therefore,

$t = 100 - 19.23$

$= 80.77$

$= 80.8$ (to 3 s.f.)

c $\mathrm{P}(s < Y < t) = 0.975 - 0.9$

$= 0.075$

7 $X \sim \mathrm{N}(80, 4^2)$

a i $\mathrm{P}(X > a) = 0.40$

$\mathrm{P}\left(z > \frac{a-80}{4}\right) = 0.4 \Rightarrow \mathrm{P}\left(z < \frac{a-80}{4}\right) = 0.6$

$\frac{a-80}{4} = 0.2534$

$a = 81.0136$

$= 81.0$ (to 3 s.f.)

ii $\mathrm{P}(X < b) = 0.5636$

$\mathrm{P}\left(z < \frac{b-80}{4}\right) = 0.5636$

$\frac{b-80}{4} = 0.1601$

$b = 80.6404$

$= 80.6$ (to 3 s.f.)

b $\mathrm{P}(b < X < a) = 0.6 - 0.5636$

$= 0.0364$

8 a $x = 0.8 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{0.8-0.8}{0.05} = 0$

b $x = 0.792 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{0.792-0.8}{0.05} = -0.16$

c $x = 0.81 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{0.81-0.8}{0.05} = 0.2$

d $x = 0.837 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{0.837-0.8}{0.05} = 0.74$

9 a $x = 154 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{154-154}{12} = 0 \Rightarrow \mathrm{P}(X < 154) = \Phi(0)$

b $x = 160 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{160-154}{12} = 0.5 \Rightarrow \mathrm{P}(X < 160) = \Phi(0.5)$

c $x = 151 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{151-154}{12} = -0.25 \Rightarrow \mathrm{P}(X > 151) = 1 - \mathrm{P}(X < 151) = 1 - \Phi(-0.25)$

d $x = 140 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{140-154}{12} = -\frac{7}{6}$

$x = 155 \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{155-154}{12} = \frac{1}{12}$

$\Rightarrow \mathrm{P}(140 < X < 155) = \mathrm{P}(X < 155) - \mathrm{P}(X < 140) = \Phi\left(\frac{1}{12}\right) - \Phi\left(-\frac{7}{6}\right)$

10 a $P(Z > z) = 0.025 \Rightarrow p = 0.025$
Using the percentage points table, $p = 0.025 \Rightarrow z = 1.96$

b Using the formula $z = \dfrac{x-\mu}{\sigma}$:

$$1.96 = \frac{x-80}{4}$$
$$x - 80 = 4 \times 1.96$$
$$x = 80 + 7.84$$
$$= 87.84$$

A score of 87.8 (3 s.f.) is needed to get on the programme.

11 a From the percentage points table, $p = 0.15 \Rightarrow z = 1.0364$
Therefore $P(Z > 1.0364) = 0.15$, hence $P(Z < -1.0364) = 0.15$, so $z = -1.0364$

b Using the formula $z = \dfrac{x-\mu}{\sigma}$:

$$-1.0364 = \frac{x-57}{2}$$
$$x - 57 = 2 \times (-1.0364)$$
$$x = 57 - 2.0728$$
$$= 54.9272$$

The size of a 'little' hat is 54.9 cm (3 s.f.).

12 a The 90th percentile corresponds to $p = 0.1$.
From the percentage points table, $p = 0.10 \Rightarrow z = 1.2816$
By the symmetry of the normal distribution, the 10th percentile is at $z = -1.2816$
So the 10% to 90% interpercentile range corresponds to $-1.2816 < z < 1.2816$

b A 'standard' light bulb should have a range of life within the above range, but for $N(1175, 56)$.

Using the formula $z = \dfrac{x-\mu}{\sigma}$ with $z = -1.2816$:

$$-1.2816 = \frac{x-1175}{56}$$
$$x - 1175 = 56 \times (-1.2816)$$
$$x = 1175 - 71.7696$$
$$= 1103.2304$$

Similarly, for $z = 1.2816$, $x = 1175 + 71.7696 = 1246.7696$.
So the range of life for a 'standard' bulb is 1103 to 1247 hours.