Chapter review

1 a

x	1	2	3	4	5
$\mathbf{P}(X = x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

b $P(3 < x \le 5) = \frac{4}{15} + \frac{5}{15}$

$= \frac{3}{5}$

2 a $\sum P(X = x) = 1$

Therefore:

$0.2 + q + 0.3 + 0.1 + 0.2 + 0.1 = 1$

$q = 0.1$

b $P(-1 \le x < 2) = 0.1 + 0.3 + 0.1$

$= 0.5$

3 a

x	1	2	3	4
$\mathbf{P}(X = x)$	$\frac{2}{26}$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{11}{26}$

b $P(2 < X \leqslant 4) = P(X = 3) + P(X = 4) = \frac{19}{26}$

4 a For a discrete uniform distribution, the probability of choosing each counter must be equal.

b i $P(X = 5) = \frac{1}{16}$

ii The prime numbers are 2, 3, 5, 7, 11 and 13

$P(X \text{ is prime}) = \frac{6}{16} = \frac{3}{8}$

iii $P(3 \leqslant X < 11) = \frac{8}{16} = \frac{1}{2}$

5 a

x	1	2	3	4	5
P(*Y* = *y*)	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{3}{k}$	$\frac{4}{k}$	$\frac{5}{k}$

$$\frac{1}{k}+\frac{2}{k}+\frac{3}{k}+\frac{4}{k}+\frac{5}{k}=1$$

$$\frac{15}{k}=1,\ k=15$$

b

x	1	2	3	4	5
P(*Y* = *y*)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}=\frac{1}{5}$	$\frac{4}{15}$	$\frac{5}{15}=\frac{1}{3}$

c $P(Y>3)=P(Y=4)+P(Y=5)=\frac{4}{15}+\frac{5}{15}=\frac{9}{15}=\frac{3}{5}$

6 a

t	0	1	2	3	4
P(*T* = *t*)	$0.75^4 = 0.316$	$0.25 \times 0.75^3 \times 4 = 0.422$	$0.25^2 \times 0.75^2 \times 6 = 0.211$	$0.25^3 \times 0.75 \times 4 = 0.0469$	$0.25^4 = 0.00391$

b $P(T<3)=P(T=0)+P(T=1)+P(T=2)=0.949$ (to 3 s.f.)

c

S	1	2	3	4	5
P(*S* = *s*)	0.25	$0.25 \times 0.75 = 0.188$	$0.25 \times 0.75^2 = 0.141$	$0.25 \times 0.75^3 = 0.105$	$0.25 \times 0.75^4 + 0.75^5 = 0.316$

d $P(S>2)=P(S=3)+P(S=4)+P(S=5)=0.563$ (to 3 s.f. using exact figures)

7 a The probability distribution for X is:

x	1	2	3	4	5	6
P(X = x)	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{6}{21}$

b $P(2 < X \leqslant 5) = P(X = 3) + P(X = 4) + P(X = 5) = \frac{3}{21} + \frac{4}{21} + \frac{5}{21} = \frac{12}{21} = 7$

c $$E(X) = 1\times\frac{1}{21} + 2\times\frac{2}{21} + 3\times\frac{3}{21} + 4\times\frac{4}{21} + 5\times\frac{5}{21} + 6\times\frac{6}{21}$$
$$= \frac{1}{21}(1+4+9+16+25+36) = \frac{91}{21} = \frac{13}{3}$$

d $$E(X^2) = 1\times\frac{1}{21} + 4\times\frac{2}{21} + 9\times\frac{3}{21} + 16\times\frac{4}{21} + 25\times\frac{5}{21} + 36\times\frac{6}{21}$$
$$= \frac{1}{21}(1+8+27+64+125+216) = \frac{441}{21} = 21$$
$$\text{Var}(X) = E(X^2) - (E(X))^2$$
$$= 21 - \left(\frac{13}{21}\right)^2 = 21 - \frac{169}{9}$$
$$= \frac{189}{9} - \frac{169}{9} = \frac{20}{9} = 2.22 \text{ (2 d.p.)}$$

e $$\text{Var}(3-2X) = \text{Var}(-2X+3)$$
$$= (-2)^2\,\text{Var}(X)$$
$$= 4\times\frac{20}{9} = \frac{80}{9} = 8.89 \text{ (2 d.p.)}$$

f $$E(X^3) = \sum x^3\, P(X = x)$$
$$= 1^3\times\frac{1}{21} + 2^3\times\frac{2}{21} + 3^3\times\frac{3}{21} + 4^3\times\frac{4}{21} + 5^3\times\frac{5}{21} + 6^3\times\frac{6}{21}$$
$$= \frac{1}{21}(1+16+81+256+625+1296)$$
$$= \frac{2275}{21} = \frac{325}{3} = 108.33 \text{ (2 d.p.)}$$

8 a Probabilities sum to 1, so:
$0.1 + 0.2 + 0.3 + r + 0.1 + 0.1 = 1$
$r = 1 - 0.8 = 0.2$

b $P(-1 \leqslant X < 2) = P(X = -1) + P(X = 0) + P(X = 1) = 0.2 + 0.3 + 0.2 = 0.7$

8 c $E(X) = -2\times0.1 + (-1)\times0.1 + 0\times0.3 + 1\times0.2 + 2\times0.1 + 3\times0.1$
$= -0.2 - 0.2 + 0.2 + 0.2 + 0.3 = 0.3$
$E(2X+3) = 2E(X) + 3 = (2\times0.3) + 3 = 3.6$

d $E(X^2) = 4\times0.1 + 1\times0.1 + 0\times0.3 + 1\times0.2 + 4\times0.1 + 9\times0.1$
$= 0.4 + 0.2 + 0.2 + 0.4 + 0.9 = 2.1$
$\text{Var}(X) = E(X^2) - (E(X))^2$
$= 2.1 - (0.3)^2 = 2.1 - 0.09 = 2.01$
$\text{Var}(2X+3) = 2^2\,\text{Var}(X)$
$= 4\times2.01 = 8.04$

9 a Probabilities sum to 1, so:
$\frac{1}{5} + b + b + \frac{1}{5} = 1$
$2b = 1 - \frac{2}{5} = \frac{3}{5}$
$b = \frac{3}{10}$

b $E(X) = 0\times\frac{1}{5} + 1\times\frac{3}{10} + 2\times\frac{5}{10} = \frac{13}{10} = 1.3$

c $E(X^2) = 0\times\frac{1}{5} + 1\times\frac{3}{10} + 4\times\frac{5}{10} = \frac{23}{10} = 2.3$
$\text{Var}(X) = E(X^2) - (E(X))^2$
$= 2.3 - 1.3^2 = 2.3 - 1.69 = 0.61$

d $P(X \leqslant 1.5) = P(X=0) + P(X=1) = \frac{1}{5} + \frac{3}{10} = 0.5$

10 a Probabilities sum to 1, so:
$k(1-0) + k(1-1) + k(2-1) + k(3-1) = 1$
$k + k + 2k = 1$
$4k = 1$
$k = \frac{1}{4} = 0.25$

10 b The probability distribution for X is:

x	0	1	2	3
P(X = x)	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$

$$E(X) = 0\times\frac{1}{4}+1\times 0+2\times\frac{1}{4}+3\times\frac{1}{2}=\frac{1}{2}+\frac{3}{2}=2$$

$$E(X^2) = 0\times\frac{1}{4}+1\times 0+4\times\frac{1}{4}+9\times\frac{1}{2}=\frac{11}{2}=5.5$$

c $\mathrm{Var}(X) = E(X^2)-(E(X))^2 = 5.5-2^2 = 5.5-4 = 1.5$

$\mathrm{Var}(2X-2) = 2^2\,\mathrm{Var}(X) = 4\times 1.5 = 6$

11 a $P(1 < X \leqslant 2) = P(X=2) = \frac{1}{8}$

b $E(X) = 0\times\frac{1}{4}+1\times\frac{1}{2}+2\times\frac{1}{8}+3\times\frac{1}{8}=\frac{1}{2}+\frac{1}{4}+\frac{3}{8}=\frac{9}{8}$

c $E(3X-1) = 3E(X)-1 = \frac{27}{8}-1=\frac{19}{8}$

d $E(X^2) = 0\times\frac{1}{4}+1\times\frac{1}{2}+4\times\frac{1}{8}+9\times\frac{1}{8}=\frac{1}{2}+\frac{1}{2}+\frac{9}{8}=\frac{17}{8}$

$\mathrm{Var}(X) = E(X^2)-(E(X))^2 = \frac{17}{8}-\left(\frac{9}{8}\right)^2=\frac{136}{64}-\frac{81}{64}=\frac{55}{64}$

e $E(\log(X+1)) = \sum \log(X+1)\,P(X=x)$

$$= \log(0+1)\times\frac{1}{4}+\log(1+1)\times\frac{1}{2}+\log(2+1)\times\frac{1}{8}+\log(3+1)\times\frac{1}{8}$$

$$= 0+\frac{1}{2}\log 2+\frac{1}{8}\log 3+\frac{1}{8}\log 4$$

$$= 0.5\times 0.30102+0.125\times 0.47712+0.125\times 0.60206$$

$$= 0.2854 \text{ (4 d.p.)}$$

12 a The probability distribution for X^2 is:

x^2	1	4	9	19
P($X^2 = x^2$)	0.4	0.2	0.1	0.3

$P(3 < X^2 < 10) = 0.2+0.1 = 0.3$

b $E(X) = 1\times 0.4+2\times 0.2+3\times 0.1+4\times 0.3 = 2.3$

12 c $\mathrm{E}(X^2) = 1\times0.4 + 4\times0.2 + 9\times0.1 + 16\times0.3 = 6.9$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$= 6.9 - (2.3)^2 = 6.9 - 5.29 = 1.61$

d $\mathrm{E}\left(\frac{3-X}{2}\right) = \frac{3}{2} - \frac{1}{2}\mathrm{E}(X) = \frac{3}{2} - \frac{2.3}{2} = \frac{0.7}{2} = 0.35$

e $\mathrm{E}\left(\sqrt{X}\right) = \sum \sqrt{X}\,\mathrm{P}(X = x)$

$= 1\times0.4 + \sqrt{2}\times0.2 + \sqrt{3}\times0.1 + \sqrt{4}\times0.3$

$= 0.4 + 0.2828 + 0.1732 + 0.6$

$= 1.4560$ (4 d.p.)

f $\mathrm{E}\left(2^{-X}\right) = \sum 2^{-X}\,\mathrm{P}(X = x)$

$= 2^{-1}\times0.4 + 2^{-2}\times0.2 + 2^{-3}\times0.1 + 2^{-4}\times0.3$

$= 0.5\times0.4 + 0.25\times0.2 + 0.125\times0.1 + 0.0625\times0.3$

$= 0.2 + 0.05 + 0.0125 + 0.01875$

$= 0.28125$

13 a Probabilities sum to 1, so:

$0.1 + p + q + 0.3 + 0.1 = 1$

$p + q = 0.5$ **(1)**

$\mathrm{E}(X) = 0.1 + 2p + 3q + 1.2 + 0.5 = 3.1$

$2p + 3q = 1.3$ **(2)**

b Multiply equation **(1)** by 2:

$2p + 2q = 1$ **(3)**

Subtract equation **(3)** from equation **(2)**

$q = 0.3$

Substitute for q in equation **(1)**

$p + 0.3 = 0.5 \Rightarrow p = 0.2$

c $\mathrm{E}(X) = 1\times0.1 + 2\times0.2 + 3\times0.3 + 4\times0.3 + 5\times0.1$

$= 0.1 + 0.4 + 0.9 + 1.2 + 0.5 = 3.1$

$\mathrm{E}(X^2) = 1\times0.1 + 4\times0.2 + 9\times0.3 + 16\times0.3 + 25\times0.1$

$= 0.1 + 0.8 + 2.7 + 4.8 + 2.5 = 10.9$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 10.9 - (3.1)^2 = 10.9 - 9.61 = 1.29$

d $\mathrm{Var}(2X - 3) = 2^2\,\mathrm{Var}(X) = 4\times1.29 = 5.16$

14 a The probability distribution for X is:

x	1	2	3	4	5
P(*X* = *x*)	k	2k	k	2k	3k

Probabilities sum to 1, so:

$k+2k+k+2k+3k=9k=1$

$k=\frac{1}{9}$

b $\mathrm{E}(X)=1k+2\times 2k+3k+4\times 2k+5\times 3k$

$=31k=\frac{31}{9}$

c $\mathrm{E}(X^2)=1k+4\times 2k+9k+16\times 2k+25\times 3k$

$=125k=\frac{125}{9}$

$\mathrm{Var}(X)=E(X^2)-(E(X))^2$

$=\frac{125}{9}-\left(\frac{31}{9}\right)^2=\frac{1125}{81}-\frac{961}{81}=\frac{164}{81}$

$=2.02$ (3 s.f.)

d $\mathrm{Var}(3-2X)=(-2)^2\,\mathrm{Var}(X)=4\times 2.02=8.1$ (1 d.p.)

15 a Probabilities sum to 1, so:

$0.1+0.3+a+b=1$

$a+b=0.6$ **(1)**

Rearrange the equation for Y to get X in terms of Y:

$3X=Y+1 \Rightarrow X=\frac{1}{3}Y+\frac{1}{3}$

$E(X)=E\left(\frac{1}{3}Y+\frac{1}{3}\right)=\frac{1}{3}E(Y)+\frac{1}{3}=\frac{1}{3}\times\frac{11}{10}+\frac{1}{3}=\frac{21}{30}=0.7$

$\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$=-0.1+a+2b=0.7$

So $a+2b=0.8$ **(2)**

Subtract equation **(1)** from equation **(2)**, which gives:

$b=0.2$

So by substituting the value for b in equation (**1**)

$a+0.2=0.6 \Rightarrow a=0.4$

b $\mathrm{E}(X^2)=1\times0.1+0\times0.3+1\times0.4+4\times0.2=1.3$

$\mathrm{Var}(X)=E(X^2)-(E(X))^2=1.3-0.7^2=1.3-0.49=0.81$

c $\mathrm{Var}(Y)=\mathrm{Var}(1-3X)=(-3)^2\,\mathrm{Var}(X)=9\times0.81=7.29$

d $\mathrm{P}(Y+2>X)=\mathrm{P}(3X-1+2>X)$

$=P(2X>-1)=P(X>-0.5)$

$=0.3+0.4+0.2=0.9$

16 a A discrete uniform distribution.

b Any distribution where all the probabilities are the same. An example is throwing a fair die.

c There are 5 possible values. So as the variable has discrete uniform distribution, each value has a $\frac{1}{5}$ (= 0.2) probability. E(X) can be found by symmetry, as the probability distribution is uniform, or by:

$\mathrm{E}(X)=0.2(0+1+2+3+4)=0.2\times10=2$

d $\mathrm{E}(X^2)=0.2(0+1+4+9+16)=0.2\times30=6$

$\mathrm{Var}(X)=E(X^2)-(E(X))^2=6-2^2=6-4=2$

Challenge

$$\mathrm{E}(X) = 1 \times \frac{1}{n} + 2 \times \frac{1}{n} + 3 \times \frac{1}{n} + 4 \times \frac{1}{n} + \ldots\ldots + n \times \frac{1}{n}$$

$$= \frac{1}{n}(1 + 2 + 3 + 4 + \ldots\ldots + n)$$

$$= \frac{1}{n}\sum_{i=1}^{n} i$$

$$= \frac{1}{n} \times \frac{n(n+1)}{2} \quad \text{(using first sum in hint)}$$

$$= \frac{n+1}{2}$$

$$\mathrm{E}(X^2) = \frac{1}{n}(1 + 4 + 9 + 16 + \ldots\ldots + n^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6} \quad \text{(using second sum in hint)}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(n+1)(2n+1)}{12} - \frac{3(n+1)^2}{12}$$

$$= \frac{4n^2 + 6n + 2 - 3n^2 - 6n - 3}{12} \quad \text{(multiplying out)}$$

$$= \frac{n^2 - 1}{12} \quad \text{(simplifying terms)}$$

$$= \frac{(n+1)(n-1)}{12} \quad \text{(factoring)}$$