Exercise 6E

1 a The probability distribution for Y where $Y = 2X - 3$ is:

y	–1	1	3	5
P(*Y* = *y*)	0.1	0.3	0.2	0.4

b $\mathrm{E}(Y) = \sum y\mathrm{P}(Y = y)$
$= -1\times0.1 + 1\times0.3 + 3\times0.2 + 5\times0.4$
$= -0.1 + 0.3 + 0.6 + 2$
$= 2.8$

c $\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$
$= 1\times0.1 + 2\times0.3 + 3\times0.2 + 4\times0.4$
$= 0.1 + 0.6 + 0.6 + 1.6$
$= 2.9$
$\mathrm{E}(2X - 3) = \mathrm{E}(Y) = 2.8$
$2\mathrm{E}(X) - 3 = 2\times2.9 - 3 = 5.8 - 3 = 2.8$
So $\mathrm{E}(2X - 3) = 2\mathrm{E}(X) - 3$

2 a The probability distribution for Y where $Y = X^3$ is:

y	–8	–1	0	1	8
P(*Y* = *y*)	0.1	0.1	0.2	0.4	0.2

b $\mathrm{E}(Y) = \sum y\mathrm{P}(Y = y)$
$= -8\times0.1 + (-1)\times0.1 + 0\times0.2 + 1\times0.4 + 8\times0.2$
$= -0.8 - 0.1 + 0 + 0.4 + 1.6$
$= 1.1$

3 a $\mathrm{E}(8X) = 8\mathrm{E}(X) = 8$

b $\mathrm{E}(X + 3) = \mathrm{E}(X) + 3 = 1 + 3 = 4$

c $\mathrm{Var}(X + 3) = \mathrm{Var}(X) = 2$

d $\mathrm{Var}(3X) = 3^2\mathrm{Var}(X) = 3^2\times2 = 9\times2 = 18$

e $\mathrm{Var}(1 - 2X) = (-2)^2\mathrm{Var}(X) = 4\times2 = 8$

f $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$
So $2 = \mathrm{E}(X^2) - 1^2$
$\Rightarrow \mathrm{E}(X^2) = 3$

4 a $\mathrm{E}(2X) = 2\mathrm{E}(X) = 2\times3 = 6$

b $\mathrm{E}(3 - 4X) = 3 - 4\mathrm{E}(X) = 3 - 4\times3 = -9$

4 c $\mathrm{E}(X^2-4X)=\mathrm{E}(X^2)-\mathrm{E}(4X)$
$=\mathrm{E}(X^2)-4\mathrm{E}(X)$
$=10-4\times 3=-2$

d $\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2=10-3^2=1$

e $\mathrm{Var}(3X+2)=3^2\mathrm{Var}(X)=9\mathrm{Var}(X)=1$

5 a $\mathrm{E}(4X)=4\mathrm{E}(X)=4\mu$

b $\mathrm{E}(2X+2)=2\mathrm{E}(X)+2=2\mu+2$

c $\mathrm{E}(2X-2)=2\mathrm{E}(X)-2=2\mu-2$

d The standard deviation of a random variable is the square root of its variance.
So if the standard deviation is σ, the variance is σ^2.
$\mathrm{Var}(2X+2)=2^2\mathrm{Var}(X)=4\sigma^2$

e $\mathrm{Var}(2X-2)=2^2\mathrm{Var}(X)=4\sigma^2$

6 a $\mathrm{E}(X)=\frac{1}{6}(1+2+3+4+5+6)=\frac{21}{6}=3.5$

b $Y=200+100X$

c $\mathrm{E}(Y)=\mathrm{E}(200+100X)=200+100\mathrm{E}(X)$
$=200+100\times 3.5=200+350=550$

7 Assume the pizzas are cylindrical and that the amount of pizza dough is given by the volume of the cylinder. The volume of a cylinder is $\pi r^2 h$. The volumes of the different sizes of pizza are then:

Size	**Small**	**Medium**	**Large**
Radius (cm)	10	15	20
Volume (cm^3)	$\pi\times 10^2\times 1=100\pi$	$\pi\times 15^2\times 1=225\pi$	$\pi\times 20^2\times 1=400\pi$

Let the amount of pizza dough be the discrete random variable V.
The probability distribution for V is:

Size	**Small**	**Medium**	**Large**
v	100π	225π	400π
P($V=v$)	$\frac{3}{10}$	$\frac{9}{20}$	$\frac{5}{20}$

$\mathrm{E}(V)=\sum v\mathrm{P}(V=v)$
$=100\pi\times\frac{3}{10}+225\pi\times\frac{9}{20}+400\pi\times\frac{5}{20}$
$=(30+101.25+100)\pi=231.25\pi$
$=726.5\text{ cm}^3$ (1 d.p.)

8 a This sample space diagram shows the 16 possible outcomes:

Difference between scores	**1**	**2**	**3**	**4**
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

Use the table to construct the probability distribution of X:

$\boldsymbol{x}$	0	1	2	3
$\mathbf{P}(\boldsymbol{X}=\boldsymbol{x})$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{2}{16}$

$$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$$
$$=0\times\tfrac{4}{16}+1\times\tfrac{6}{16}+2\times\tfrac{4}{16}+3\times\tfrac{2}{16}=\tfrac{20}{16}=\tfrac{5}{4}=1.25$$
$$\mathrm{E}(X^2)=\sum x^2\mathrm{P}(X=x)$$
$$=0^2\times\tfrac{4}{16}+1^2\times\tfrac{6}{16}+2^2\times\tfrac{4}{16}+3^2\times\tfrac{2}{16}=\tfrac{40}{16}=\tfrac{5}{2}=2.5$$
$$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$$
$$=2.5-(1.25)^2=2.5-1.5625=0.9375$$

b The probability distribution for Y where $Y=2^X$ is:

$\boldsymbol{x}$	0	1	2	3
$\boldsymbol{y}$	1	2	4	8
$\mathbf{P}(\boldsymbol{Y}=\boldsymbol{y})$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

$$\mathrm{E}(Y)=\sum y\mathrm{P}(Y=y)$$
$$=1\times\tfrac{1}{4}+2\times\tfrac{3}{8}+4\times\tfrac{1}{4}+8\times\tfrac{1}{8}=\tfrac{24}{8}=3$$

The probability distribution for Z where $Z=\dfrac{4X+1}{2}$ is:

$\boldsymbol{x}$	0	1	2	3
$\boldsymbol{z}$	0.5	2.5	4.5	6.5
$\mathbf{P}(\boldsymbol{Z}=\boldsymbol{z})$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

$$\mathrm{E}(Z)=\sum z\mathrm{P}(Z=z)$$
$$=0.5\times\tfrac{1}{4}+2.5\times\tfrac{3}{8}+4.5\times\tfrac{1}{4}+6.5\times\tfrac{1}{8}$$
$$=0.125+0.9375+1.125+0.8125=3$$

So E(Y) = E(Z) = 0.3

8 c $\mathrm{E}(Z^2) = \sum z^2 \mathrm{P}(Z = z)$

$$= (0.5)^2 \times \tfrac{1}{4} + (2.5)^2 \times \tfrac{3}{8} + (4.5)^2 \times \tfrac{1}{4} + (6.5)^2 \times \tfrac{1}{8}$$

$$= 0.0625 + 2.34375 + 5.0625 + 5.28125 = 12.75$$

$$\mathrm{Var}(Z) = \mathrm{E}(Z^2) - (\mathrm{E}(Z))^2 = 12.75 - 3^2 = 3.75$$

Alternatively:

$$Z = \frac{4X+1}{2} \quad \Rightarrow \quad Z = 2X + 0.5$$

$$\mathrm{Var}(Z) = 2^2\,\mathrm{Var}(X) = 4\mathrm{Var}(X) = 4 \times 0.9375 = 3.75$$

Challenge

$$\begin{aligned}\text{LHS} &= \mathrm{E}((X - \mathrm{E}(X))^2) \\ &= \mathrm{E}(X^2 - 2X\mathrm{E}(X) + (\mathrm{E}(X))^2) \\ &= \mathrm{E}(X^2) - \mathrm{E}(2X\mathrm{E}(X)) + \mathrm{E}((\mathrm{E}(X))^2)\end{aligned}$$

so $\mathrm{E}(2X\mathrm{E}(X)) = 2\mathrm{E}(X)\mathrm{E}(X) = 2(\mathrm{E}(X))^2$

and $\mathrm{E}((\mathrm{E}(\mathrm{X}))^2) = (\mathrm{E}(\mathrm{X}))^2$

Substituting back into the equation above gives

$$\begin{aligned}\text{LHS} &= \mathrm{E}(X^2) - \mathrm{E}(2X\mathrm{E}(X)) + \mathrm{E}((\mathrm{E}(X))^2) \\ &= \mathrm{E}(X^2) - 2(\mathrm{E}(X))^2 + (\mathrm{E}(X))^2 \\ &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\ &= \text{RHS}\end{aligned}$$