Exercise 6D

1 a By symmetry $\text{E}(X)=1$

Alternatively, use $\text{E}(X)=\sum x\,\text{P}(X=x)$

$\text{E}(X)=\frac{1}{5}(-1+0+1+2+3)=\frac{1}{5}\times 5=1$

b $\text{E}(X^2)=\sum x^2\,\text{P}(X=x)$

$\text{E}(X^2)=\frac{1}{5}(1+0+1+4+9)=\frac{1}{5}\times 15=3$

$\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2$

$=3-1^2=2$

2 a $\text{E}(X)=\sum x\,\text{P}(X=x)$

$=1\times\frac{1}{3}+2\times\frac{1}{2}+3\times\frac{1}{6}$

$=\frac{1}{3}+1+\frac{1}{2}=\frac{11}{6}=1.833$ (3 d.p.)

$\text{E}(X^2)=\sum x^2\,\text{P}(X=x)$

$=1\times\frac{1}{3}+4\times\frac{1}{2}+9\times\frac{1}{6}$

$=\frac{1}{3}+2+\frac{3}{2}=\frac{23}{6}$

$\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2$

$=\frac{23}{6}-\left(\frac{11}{6}\right)^2=\frac{138}{36}-\frac{121}{36}=\frac{17}{36}=0.472$ (3 d.p.)

b $\text{E}(X)=\sum x\,\text{P}(X=x)$

$=-1\times\frac{1}{4}+0\times\frac{1}{2}+1\times\frac{1}{4}=0$ (or derive answer by symmetry)

$\text{E}(X^2)=\sum x^2\,\text{P}(X=x)$

$=1\times\frac{1}{4}+0\times\frac{1}{2}+1\times\frac{1}{4}=\frac{1}{2}=0.5$

$\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2=0.5-0^2=0.5$

c $\text{E}(X)=\sum x\,\text{P}(X=x)$

$=(-2)\times\frac{1}{3}+(-1)\times\frac{1}{3}+1\times\frac{1}{6}+2\times\frac{1}{6}$

$=-1+\frac{1}{2}=-\frac{1}{2}=-0.5$

$\text{E}(X^2)=\sum x^2\,\text{P}(X=x)$

$=4\times\frac{1}{3}+1\times\frac{1}{3}+1\times\frac{1}{6}+4\times\frac{1}{6}$

$=\frac{5}{3}+\frac{5}{6}=\frac{15}{6}=2.5$

$\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2=2.5-(0.5)^2=2.5-0.25=2.25$

3 The probability distribution for Y is:

y	1	2	3	4	5	6	7	8
$\text{P}(Y=y)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

$\text{E}(Y)=\frac{1}{8}(1+2+3+4+5+6+7+8)=\frac{1}{8}\times 36=4.5$

$\text{E}(Y^2)=\frac{1}{8}(1+4+9+16+25+36+49+64)=\frac{1}{8}\times 204=25.5$

$\text{Var}(Y)=\text{E}(Y^2)-(\text{E}(Y))^2=25.5-(4.5)^2=25.5-20.25=5.25$

4 a This sample space diagram shows the 36 possible outcomes:

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Use the table to construct the probability distribution of S:

s	2	3	4	5	6	7	8	9	10	11	12
P($S = s$)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b
$$E(S) = \tfrac{1}{36}(2\times1+3\times2+4\times3+5\times4+6\times5+7\times6+8\times5+9\times4+10\times3+11\times2+12\times1)$$
$$= \tfrac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$$
$$= \tfrac{252}{36} = 7$$

The answer can also be derived by symmetry.

c
$$\mathrm{E}(S^2) = \tfrac{1}{36}(4+9\times2+16\times3+25\times4+36\times5+49\times6+64\times5+81\times4+100\times3+121\times3+144)$$
$$= \tfrac{1}{36}(4+18+48+100+180+294+320+324+300+242+144)$$
$$= \tfrac{1974}{36} = 54.833 \text{ (3 d.p.)}$$
$$\mathrm{Var}(S) = \mathrm{E}(S^2) - (\mathrm{E}(S))^2$$
$$= \tfrac{1974}{36} - (7)^2 = \tfrac{1974}{36} - 49 = \tfrac{1974-1764}{36}$$
$$= \tfrac{210}{36} = 5.833 \text{ (3 d.p.)}$$

d Standard deviation $= \sqrt{5.8333} = 2.415$ (3 d.p.)

5 a This sample space diagram shows the 16 possible outcomes:

Difference between scores	**1**	**2**	**3**	**4**
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

Use the table to construct the probability distribution of D:

d	0	1	2	3
P(*D* = *d*)	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{2}{16}$

Simplify the probabilities:

d	0	1	2	3
P(*D* = *d*)	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

b $\mathrm{E}(D) = 0\times\frac{1}{4}+1\times\frac{3}{8}+2\times\frac{1}{4}+3\times\frac{1}{8} = \frac{10}{8} = \frac{5}{4} = 1.25$

c $\mathrm{E}(D^2) = 0\times\frac{1}{4}+1\times\frac{3}{8}+4\times\frac{1}{4}+9\times\frac{1}{8} = \frac{20}{8} = \frac{5}{2} = 2.5$

$\mathrm{Var}(D) = \mathrm{E}(D^2) - (\mathrm{E}(D))^2$

$= 2.5 - (1.25)^2 = 2.5 - 1.5625 = 0.9375$

Alternatively, in fractional form

$\mathrm{Var}(D) = \frac{5}{2} - \left(\frac{5}{4}\right)^2 = \frac{5}{2} - \frac{25}{16} = \frac{40}{16} - \frac{25}{16} = \frac{15}{16}$

6 a $\mathrm{P}(\text{heads on first spin}) = \frac{1}{2} \Rightarrow \mathrm{P}(T=1) = \frac{1}{2}$

$\mathrm{P}(\text{tails on first spin, heads on second spin}) = \frac{1}{2}\times\frac{1}{2} \Rightarrow \mathrm{P}(T=2) = \frac{1}{4}$

$\mathrm{P}(T=3) = 1 - \left(\mathrm{P}(T=1)+\mathrm{P}(T=2)\right) = 1 - \left(\frac{1}{2}+\frac{1}{4}\right) = \frac{1}{4}$

Alternatively note that

$\mathrm{P}(T=3) = P(\text{heads, heads, tails}) + P(\text{heads, heads, heads})$

$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$

b $\mathrm{E}(T) = 1\times\frac{1}{2}+2\times\frac{1}{4}+3\times\frac{1}{4} = \frac{7}{4} = 1.75$

$\mathrm{E}(T^2) = 1\times\frac{1}{2}+4\times\frac{1}{4}+9\times\frac{1}{4} = \frac{15}{4} = 3.75$

$\mathrm{Var}(T) = \frac{15}{4} - \left(\frac{7}{4}\right)^2 = \frac{60}{16} - \frac{49}{16} = \frac{11}{16} = 0.6875$

7 a $\mathrm{E}(X) = \sum x\mathrm{P}(X=x) = a + 2b + 3a = 4a + 2b$

7 **b** $\sum p(x) = 1$, so $2a + b = 1$ **(1)**

As $\mathrm{E}(X) = 4a + 2b = 2(2a + b)$

$\Rightarrow \mathrm{E}(X) = 2$

$\mathrm{E}(X^2) = a + 4b + 9a = 10a + 4b$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$= 10a + 4b - 2^2 = 10a + 4b - 4$

As Var(X) = 0.75, this gives

$10a + 4b = 4.75$ **(2)**

Multiply equation (**1**) by 4 to give

$8a + 4b = 4$ **(3)**

Subtract (**3**) from (**2**)

$2a = 4.75 - 4 = 0.75 \Rightarrow a = 0.375$

Substitute value of a in (**1**)

$0.75 + b = 1 \Rightarrow b = 0.25$