Exercise 6C

1 a The probability distribution for X^2 is:

x	2	4	6	8
$\mathrm{P}(X=x)$	0.3	0.3	0.2	0.2
x^2	4	16	36	64
$\mathrm{P}(X^2=x^2)$	0.3	0.3	0.2	0.2

Note that for this variable $\mathrm{P}(X=x)=\mathrm{P}(X^2=x^2)$ as X only takes positive values.

$$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$$
$$=2\times0.3+4\times0.3+6\times0.2+8\times0.2=4.6$$

$$\mathrm{E}(X^2)=\sum x^2\mathrm{P}(X=x)$$
$$=4\times0.3+16\times0.3+36\times0.2+64\times0.2=26$$

b The probability distribution for X is:

x	–2	–1	1	2
$\mathrm{P}(X=x)$	0.1	0.4	0.1	0.4
x^2	4	1	1	4

In this case, X can take negative values, so calculate the values of $\mathrm{P}(X^2=x^2)$.

$$\mathrm{P}(X^2=1)=\mathrm{P}(X=-1)+\mathrm{P}(X=1)=0.4+0.1=0.5$$
$$\mathrm{P}(X^2=4)=\mathrm{P}(X=-2)+\mathrm{P}(X=2)=0.1+0.4=0.5$$

The probability distribution for X^2 is:

x^2	1	4
$\mathrm{P}(X^2=x^2)$	0.5	0.5

$$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$$
$$=-2\times0.1+(-1)\times0.4+1\times0.1+2\times0.8$$
$$=0.3$$

$$\mathrm{E}(X^2)=\sum x^2\mathrm{P}(X=x)$$
$$=4\times0.1+1\times0.4+1\times0.1+4\times0.4$$
$$=2.5$$

Using the X^2 distribution to calculate $\mathrm{E}(X^2)$ gives the same result

$$\mathrm{E}(X^2)=\sum x^2\mathrm{P}(X^2=x^2)=1\times0.5+4\times0.5=2.5$$

2 $\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$
$= (1 \times 0.1) + (2 \times 0.1) + (3 \times 0.1) + (4 \times 0.2) + (5 \times 0.4) + (6 \times 0.1)$
$= 0.1 + 0.2 + 0.3 + 0.8 + 2.0 + 0.6$
$= 4$

$\mathrm{E}(X^2) = \sum x^2\,\mathrm{P}(X = x)$
$= (1 \times 0.1) + (4 \times 0.1) + (9 \times 0.1) + (16 \times 0.2) + (25 \times 0.4) + (36 \times 0.1)$
$= 0.1 + 0.4 + 0.9 + 3.2 + 10 + 3.6$
$= 18.2$

3 **a** The probability distribution for X is:

x	2	3	6
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

The probability distribution for X^2 is:

x^2	4	9	36
$\mathrm{P}(X^2 = x^2)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

b $\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$
$= 2 \times \frac{1}{2} + 3 \times \frac{1}{3} + 6 \times \frac{1}{6}$
$= 1 + 1 + 1$
$= 3$

$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x)$
$= 4 \times \frac{1}{2} + 9 \times \frac{1}{3} + 36 \times \frac{1}{6}$
$= 11$

c $(\mathrm{E}(X))^2 = 3^2 = 9$ and $\mathrm{E}(X^2) = 11$ from part **b**
So $(\mathrm{E}(X))^2$ does not equal $\mathrm{E}(X^2)$

4 **a** The probability distribution for X is:

x	1	2	3	4	5
$\mathrm{P}(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

b $\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$

$$=1\times\tfrac{1}{2}+2\times\tfrac{1}{4}+3\times\tfrac{1}{8}+4\times\tfrac{1}{16}+5\times\tfrac{1}{16}$$
$$=\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{6}{16}+\tfrac{9}{16}=\tfrac{31}{16}$$
$$=1.9375$$

$\mathrm{E}(X^2)=\sum x^2\mathrm{P}(X=x)$

$$=1^2\times\tfrac{1}{2}+2^2\times\tfrac{1}{4}+3^2\times\tfrac{1}{8}+4^2\times\tfrac{1}{16}+5^2\times\tfrac{1}{16}$$
$$=1\times\tfrac{1}{2}+4\times\tfrac{1}{4}+9\times\tfrac{1}{8}+16\times\tfrac{1}{16}+25\times\tfrac{1}{16}$$
$$=\tfrac{83}{16}$$
$$=5.1875$$

c $(\mathrm{E}(X))^2=(1.9375)^2=3.7539$ (4 d.p.)

So $(\mathrm{E}(X))^2$ does not equal $\mathrm{E}(X^2)$

5 The probabilities add up to 1, so

$$0.1+a+b+0.2+0.1=1$$
$$a+b=0.6 \qquad \textbf{(1)}$$

$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)=2.9$, so

$$(1\times0.1)+(2\times a)+(3\times b)+(4\times0.2)+(5\times0.1)=2.9$$
$$0.1+2a+3b+0.8+0.5=2.9$$
$$2a+3b=1.5 \qquad \textbf{(2)}$$

Multiply (**1**) by 2

$$2a+2b=1.2 \qquad \textbf{(3)}$$

Subtract equation (**3**) from (**1**) to give

$b=0.3$

Substitute the value of b in equation (**3**) to obtain

$$2a+0.6=1.2$$
$$\Rightarrow a=0.3$$

6 The probability distribution for X is:

x	–2	–1	0	5
P(*X* = *x*)	$3a$	$2a$	a	b

$\mathrm{E}(X) = \sum x\mathrm{P}(X = x) = 1.2$, so

$1.2 = -2\times 3a - 1\times 2a + 0\times a + 5\times b$

$1.2 = -6a - 2a + 5b$

$1.2 = -8a + 5b$ **(1)**

$\sum P(X = x) = 1$, so

$1 = 3a + 2a + a + b$

$1 = 6a + b$ **(2)**

(**2**)$\times 5 \Rightarrow 5 = 30a + 5b$ **(3)**

(**3**) – (**1**) $\Rightarrow 3.8 = 38a \Rightarrow a = 0.1$

Substituting for a in equation (**2**) gives

$b = 1 - 6a = 1 - 0.6 = 0.4$

So the full solution is

$a = 0.1,\quad b = 0.4$

7 Suppose the probability distribution for X is:

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	a	b

$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)=4.1$, so

$$4.1=1\times\tfrac{1}{8}+2\times\tfrac{1}{8}+3\times\tfrac{1}{8}+4\times\tfrac{1}{8}+5\times a+6\times b$$

$$4.1=\tfrac{10}{8}+5a+6b$$

$$2.85=5a+6b \qquad \textbf{(1)}$$

$\sum P(X=x)=1$, so

$$1=\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+a+b$$

$$0.5=a+b \qquad \textbf{(2)}$$

$$\textbf{(2)}\times 5 \quad \Rightarrow \quad 2.5=5a+5b \qquad \textbf{(3)}$$

$$\textbf{(1)}-\textbf{(3)} \quad \Rightarrow \quad 0.35=b$$

Substituting for b in equation **(2)** gives

$$0.5=a+0.35 \Rightarrow a=0.15$$

So the full solution is

$$a=0.15=\tfrac{3}{20}, \quad b=0.35=\tfrac{7}{20}$$

So the full probability distribution for X is:

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{20}$	$\frac{7}{20}$

8 P(faulty) = 0.02
Profit on working phone cover is \$3.
Loss on faulty phone cover is \$8.

Profit per phone $=\dfrac{49\times 3-1\times 8}{50}=\$2.78$

Challenge

When three dice are thrown there are $6\times6\times6=216$ outcomes.

There is only one way of the number 1 being the highest score on the three dice and that is 1, 1, 1.

To achieve the highest score of 2, each dice must be either 1 or 2. So there are $2\times2\times2=8$ ways for the highest score on three dice to be no more than 2. But one of those is 1, 1, 1, which gives a highest score of 1 so this needs to be subtracted to leave 7 possible ways for a highest score of 2.

To achieve the highest score of 3, each dice must be either 1 or 2 or 3. So there are $3\times3\times3=27$ ways for the highest score on three dice to be no more than 3. But one of those is 1, 1, 1, which gives a highest score of 1 and there are 7 possible ways for a highest score of 2 so these both need to be subtracted to give 19 ways of getting a highest score of 3.

Using this approach, this is the number of ways of getting each highest score:

Highest score on the three dice	**Working**	**Number of ways**
1	1, 1, 1	1
2	$2\times2\times2-1$	7
3	$3\times3\times3-7-1$	19
4	$4\times4\times4-19-7-1$	37
5	$5\times5\times5-37-19-7-1$	61
6	$6\times6\times6-61-37-19-7-1$	91

Converting the number of ways into probabilities gives:

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{216}$	$\frac{7}{216}$	$\frac{19}{216}$	$\frac{37}{216}$	$\frac{61}{216}$	$\frac{91}{216}$

$$\begin{aligned}\mathrm{E}(X)&=\sum x\mathrm{P}(X=x)\\&=1\times\tfrac{1}{216}+2\times\tfrac{7}{216}+3\times\tfrac{19}{216}+4\times\tfrac{37}{216}+5\times\tfrac{61}{216}+6\times\tfrac{91}{216}\\&=\tfrac{1071}{216}=\tfrac{119}{24}=4.9583\ (4\text{ d.p.})\end{aligned}$$