Exercise 4D

1 **a** This is the set of anything not in set B but in set A. So the shaded region consists of the part of A which does not intersect with B, i.e. $A \cap B'$.

b The shaded region includes all of B and the region outside of A and B, i.e. $B \cup A'$.

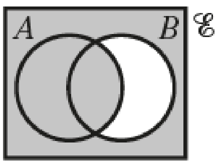
c There are two regions to describe. The first is the intersection of A and B, i.e. $A \cap B$ and the second is everything that is not in either A or B, i.e. $A' \cap B'$. Therefore the shaded region is $(A \cap B) \cup (A' \cap B')$.

d The shaded region is anything that is in A and B and C, i.e. $A \cap B \cap C$.

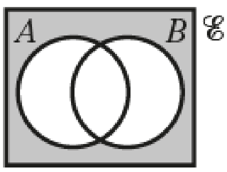
e The shaded region is anything that is either in A or B or C, i.e. $A \cup B \cup C$.

f The shaded region is anything that is either in A or B but is not in C. So the shaded region consists of the part of $A \cup B$ which does not intersect with C, i.e. $(A \cup B) \cap C'$.

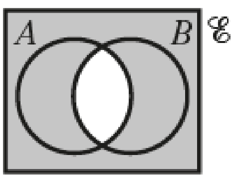
2 **a** Shade set A. The set B' consists of the region outside of A and B and the region inside A that does not intersect B. Therefore $A \cup B'$ is the region consisting of both these regions.



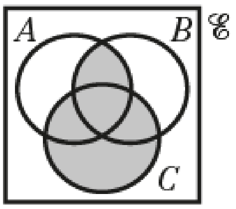
b Since this is an intersection, the region must satisfy both conditions. The first is to be in A'. This consists of two regions: one inside B and not in $A \cap B$; and one outside of A and B. The second condition is to be in B'. Again, this consists of two regions: one inside A and not in $A \cap B$; and one outside of A and B. Therefore $A' \cap B'$ is the region outside of A and B (since this region was in both A' and B'). One way to help picture this is to shade the regions A' and B' differently (either with different colours or using a different pattern for each). The intersection is then the region that includes both colours or patterns.



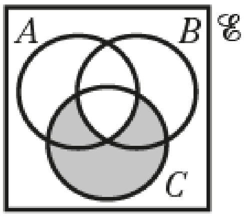
c In order to describe $(A \cap B)'$ it is sensible to first describe $A \cap B$. This is the single region which is included in both A and B. The complement is then everything *except* this region.



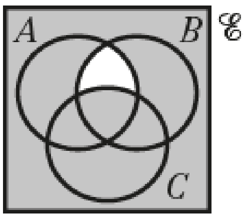
3 **a** The set $(A \cap B) \cup C$ is the union of the sets $A \cap B$ and C. On the blank diagram, the set $A \cap B$ consists of the two regions that are both contained within A and B. The remaining regions within set C can then be shaded in.



b First describe $A' \cup B'$. The set $A' \cup B'$ is everything apart from $A \cap B$. So the intersection of $A' \cup B'$ and C is everything in C apart from that part of C that intersects $A \cap B$.



c First describe $A \cap B \cap C'$. Brackets have not been included since for any sets X, Y and Z $(X \cap Y) \cap Z = X \cap (Y \cap Z)$. The intersection of $A \cap B$ and C' is the region within $A \cap B$ that does not intersect C. Therefore $(A \cap B \cap C')'$ is everything *except* this region.



4 **a** K is the event 'the card chosen is a king'.

$$P(K) = \frac{4}{52} = \frac{1}{13}$$

b C is the event 'the card chosen is a club'.

$$P(C) = \frac{1}{4}$$

c $C \cap K$ is the event 'the card chosen is the king of clubs'.

$$P(C \cap K) = \frac{1}{52}$$

d $C \cup K$ is the event 'the card chosen is a club or a king or both'.

$$P(C \cup K) = \frac{16}{52} = \frac{4}{13}$$

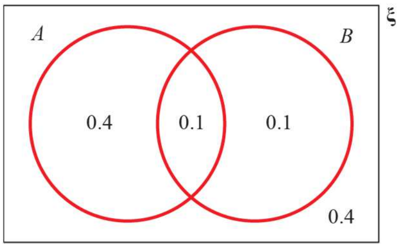
4 e C' is the event 'the card chosen is a not a club'.

$$\mathrm{P}(C') = \frac{3}{4}$$

f $K' \cap C$ is the event 'the card chosen is not a king and is a club'.

$$\mathrm{P}(K' \cap C) = \frac{12}{52} = \frac{3}{13}$$

5 Use the information in the question to draw a Venn diagram that will help in answering each part.



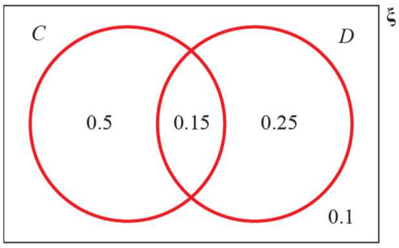
a $A \cup B$ is the region contained by sets A and B. So $\mathrm{P}(A \cup B) = 0.4 + 0.1 + 0.1 = 0.6$

b B' is the region that is not in set B. $\mathrm{P}(B') = 0.8$

c $A \cap B'$ is the region inside set A but outside set B. $\mathrm{P}(A \cap B') = 0.4$

d $A \cup B'$ is the region inside set A and the region outside set B, i.e. everything but the region inside set B that is not also in set A. $\mathrm{P}(A \cup B') = 0.4 + 0.1 + 0.4 = 0.9$

6 Use the information in the question to draw a Venn diagram that will help in answering each part.



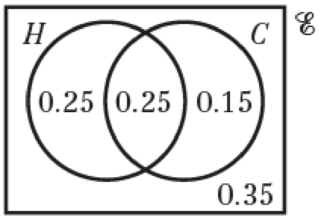
a $C' \cap D$ is the region inside set D but outside set C. $\mathrm{P}(C' \cap D) = 0.25$

b $C \cap D'$ is the region inside set C but outside set D. $\mathrm{P}(C \cap D') = 0.5$

c $\mathrm{P}(C) = 0.65$

d $C' \cup D'$ is the region outside set C and the region outside set D, i.e. everything but the region that is in both sets C and D. $\mathrm{P}(C' \cup D') = 0.85$

7 a

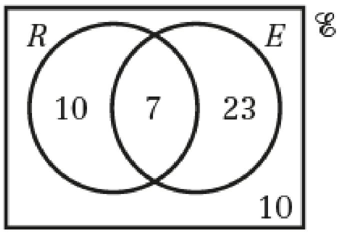


b i $P(H \cup C)$ means that either one of $H \cap C'$, $H \cap C$ or $H' \cap C$ occurs. Alternatively,
$P(H \cup C) = P(H) + P(C) - P(H \cap C) = 0.5 + 0.4 - 0.25 = 0.65$

ii $H' \cap C$ is the region inside set C but outside set H. $P(H' \cap C) = 0.15$

iii $H \cup C'$ is the region inside set H and the region outside set C, i.e. everything but the region inside set C that is not also in set H. $P(H \cup C') = 0.25 + 0.25 + 0.35 = 0.85$

8 a Only the possible outcomes of the two events need to considered, and so the Venn diagram should consist of two circles, one labelled 'R' for red and one labelled 'E' for even. They should intersect.



b i Note that $n(R \cup E) = n(R) + n(E) - n(R \cap E)$
$n(R \cap E) = n(R) + n(E) - n(R \cup E)$
$\Rightarrow n(R \cap E) = 17 + 30 - 40 = 7$

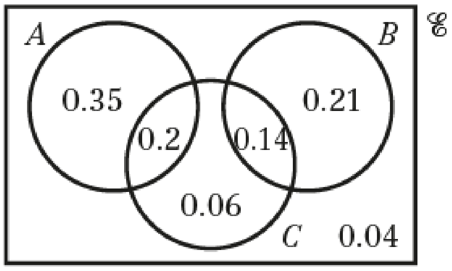
ii The region $R' \cap E'$ lies outside of both R and E.
Since there are 50 counters, $n(R' \cap E') = 50 - n(R \cup E) = 50 - 40 = 10$

So $P(R' \cap E') = \frac{10}{50} = \frac{1}{5} = 0.2$

iii From part **b i** $n(R \cap E) = 7$, so $n(R \cap E)' = 50 - 7 = 43$

So $P((R \cap E)') = \frac{43}{50} = 0.86$

9 a Since A and B are mutually exclusive, $P(A \cap B) = 0$ and they need no intersection on the Venn diagram. From the question, $P(A \cap C) = 0.2$ and so this can immediately be added to the diagram. Since B and C are independent, $P(B \cap C) = P(B) \times P(C) = 0.35 \times 0.4 = 0.14$ and this can also be added to the diagram. The remaining region in B must be $P(B) - P(B \cap C) = 0.35 - 0.14 = 0.21$, the remaining region for A must be $P(A) - P(A \cap C) = 0.55 - 0.2 = 0.35$ and the remaining region for C must be $P(C) - P(A \cap C) - P(B \cap C) = 0.4 - 0.2 - 0.14 = 0.06$. This means that the region outside of A, B and C must be 1 – 0.35 – 0.2 – 0.21 – 0.14 – 0.06 = 0.04.



b i The set $A' \cap B'$ must be outside of A and outside of B. These two regions are labelled 0.06 and 0.04. Therefore $P(A' \cap B') = 0.06 + 0.04 = 0.1$

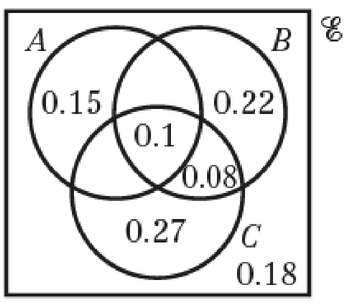
ii The region $B \cap C'$ is the region inside set B but outside set C, it is labelled 0.21 on the Venn diagram and is disjoint from A. Therefore $P(A \cup (B \cap C')) = P(A) + 0.21 = 0.55 + 0.21 = 0.76$

iii Since $A \cap C$ consists of a single region, $(A \cap C)'$ consists of everything in the diagram except for that region. But B' includes the region $A \cap C$ and so $(A \cap C)' \cup B$ includes everything in the diagram, and so $P((A \cap C)' \cup B') = 1$

10 a Start with a Venn diagram with all possible intersections. Then find the region $A \cap B \cap C$, which is at the centre of the diagram, and label it 0.1.

Now, since A and B are independent, $P(A \cap B) = P(A) \times P(B) = 0.25 \times 0.4 = 0.1$, and as B and C are independent $P(B \cap C) = P(B) \times P(C) = 0.4 \times 0.45 = 0.18$. Use these results to find values for the other intersections. $P(A \cap B \cap C') = P(A \cap B) - P(A \cap B \cap C) = 0.1 - 0.1 = 0$;
$P(B \cap C \cap A') = P(B \cap C) - P(A \cap B \cap C) = 0.18 - 0.1 = 0.08$; and $P(A \cap C \cap B') = 0$ is given in the question.

Now find values for the remaining parts of the diagram. For example,
$P(A \cap B' \cap C') = P(A) - P(A \cap B \cap C') - P(A \cap C \cap B') - P(A \cap B \cap C) = 0.25 - 0 - 0 - 0.1 = 0.15$
Similarly, $P(B \cap A' \cap C') = 0.4 - 0.1 - 0.08 = 0.22$ and $P(C \cap A' \cap B') = 0.45 - 0.1 - 0.08 = 0.27$
Finally calculate the region outside sets A, B and C,
$P(A \cup B \cup C)' = 1 - 0.15 - 0.1 - 0.22 - 0.08 - 0.27 = 0.18$

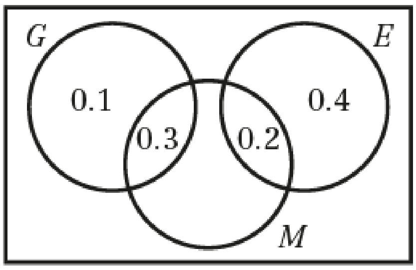


10 b i There are several ways to work out the regions that comprise the set $A' \cap (B' \cup C)$. One way is to determine, for each region, whether it lies in A' and $B' \cup C$. Alternatively, find the regions within A' (there are four) and then note that only one of these does not lie in $B' \cup C$. Summing the three remaining probabilities yields $\mathrm{P}(A' \cap (B' \cup C)) = 0.27 + 0.08 + 0.18 = 0.53$

ii The required region must be contained within C. Three of the four regions in C also lie in $A \cup B$, summing the probabilities yields $\mathrm{P}((A \cup B) \cap C) = 0 + 0.1 + 0.08 = 0.18$

c $\mathrm{P}(A') = 1 - P(A) = 0.75$, $\mathrm{P}(C) = 0.45$ and, from the Venn diagram, $\mathrm{P}(A' \cap C) = 0.08 + 0.27 = 0.35$. Since $\mathrm{P}(A') \times \mathrm{P}(C) = 0.75 \times 0.45 = 0.3375 \neq 0.35$, the events A' and C are not independent.

11 a Since $\mathrm{P}(G \cap E) = 0$, it follows that $\mathrm{P}(M \cap G \cap E) = 0$. So $\mathrm{P}(M \cap G \cap E') = \mathrm{P}(M \cap G) = 0.3$ and $\mathrm{P}(G \cap M') = \mathrm{P}(G) - \mathrm{P}(G \cap M) = 0.4 - 0.3 = 0.1$. This only accounts for 40% of the book club, 60% is unaccounted for, but P(E) = 0.6, so this 60% read epic fiction. So all the remaining members who read murder mysteries must also read epic fiction. Therefore $\mathrm{P}(M \cap E' \cap G') = 0$, $\mathrm{P}(M \cap E \cap G') = \mathrm{P}(M) - \mathrm{P}(M \cap G) = 0.5 - 0.3 = 0.2$, and $\mathrm{P}(E \cap M' \cap G') = 0.6 - 0.2 = 0.4$.



b i $\mathrm{P}(M \cup G) = \mathrm{P}(M \cup G \cup E) - \mathrm{P}(E \cap M' \cap G') = 1 - 0.4 = 0.6$

ii In this case $\mathrm{P}((M \cap G) \cup (M \cap E)) = \mathrm{P}((M \cap G \cap E') \cup (M \cap G' \cap E))$ and so the required probability is $\mathrm{P}(M \cap G \cap E') + \mathrm{P}(M \cap G' \cap E) = 0.3 + 0.2 = 0.5$

c $\mathrm{P}(G') = 0.6$, $\mathrm{P}(M) = 0.5$ and so $\mathrm{P}(G') \times \mathrm{P}(M) = 0.6 \times 0.5 = 0.3$. Since $\mathrm{P}(G' \cap M) = 0.2$, the events are not independent.

12 a Since A and B are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) = x \times y = xy$

b $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = x + y - xy$

c $\mathrm{P}(A \cup B') = \mathrm{P}(A) + \mathrm{P}(A' \cap B')$ and since
$\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B) = 1 - (x + y - xy) = 1 - x - y + xy$ this means
$\mathrm{P}(A \cup B') = \mathrm{P}(A) + 1 - x - y + xy = x + 1 - x - y + xy = 1 - y + xy$

Challenge

a Use that the events are independent.

$$\begin{aligned}P(A\cap B\cap C) &= P((A\cap B)\cap C)\\ &= P(A\cap B)\times P(C)\\ &= P(A)\times P(B)\times P(C)\\ &= xyz\end{aligned}$$

b Using similar logic to the identity $P(A\cup B)=P(A)+P(B)-P(A\cap B)$, build up to the correct expression. First, x represents one circle and its intersections with the other two circles being shaded. Then $x + y - xy$ represents two circles and their intersections with the third being shaded. Finally $x + y - xy + z - xz - yz$ represents all three circles shaded except for where all three intersect. From part **a**, the final expression is therefore $x + y - xy + z - xz - yz + xyz$.

An alternative approach is to start by considering $A\cup B$
$P(A\cup B)=P(A)+P(B)-P(A\cap B)=x+y-xy$

Now find the union of $A\cup B$ and C
$P(A\cup B\cup C)=P(A\cup B)+P(C)-P((A\cup B)\cap C)=x+y+z-xy-P((A\cup B)\cap C)$ **(1)**

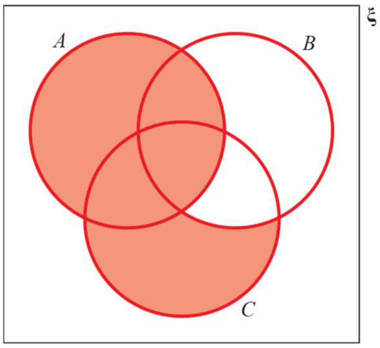
$(A\cup B)\cap C$ consists of the intersections of C with just A, with just B and with both A and B
So $(A\cup B)\cap C=(C\cap A\cap B')+(C\cap B\cap A')+(A\cap B\cap C)$

Consider the probabilities of each of these three regions in turn
$P(A\cap B\cap C)=xyz$ from part **a**
$P(C\cap A\cap B')=P(C\cap A)-P(A\cap B\cap C)=xz-xyz$
$P(C\cap B\cap A')=P(C\cap B)-P(A\cap B\cap C)=yz-xyz$
So $P(A\cup B)\cap C=xz-xyz+yz-xyz+xyz=xz+yz-xyz$ **(2)**

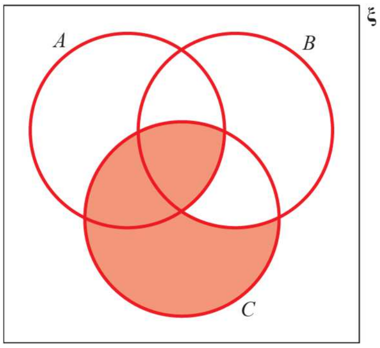
Now substitute the result for $P(A\cup B)\cap C$ from equation (**2**) into equation (**1**). This gives
$P(A\cup B\cup C)=x+y+z-xy-xz-yz+xyz$

Challenge

c First understand the region on a Venn diagram. The set $A \cup B'$ corresponds to the shaded regions:



Therefore the set $(A \cup B') \cap C$ corresponds to the shaded regions:



The unshaded part of C is the region $C \cap B \cap A'$

$\mathrm{P}(C \cap B \cap A') = P(C \cap B) - P(A \cap B \cap C) = yz - xyz$

So $\mathrm{P}((A \cup B') \cap C) = \mathrm{P}(C) - \mathrm{P}(C \cap B \cap A') = z - yz + xyz$